November 3, 2002

Particle Size Distributions: Theory and Application to Aerosols, Clouds, and Soils

by Charlie Zender
University of California at Irvine

Department of Earth System Science
University of California
Irvine, CA 92697-3100

zender@uci.edu
Voice: (949) 824-2987
Fax: (949) 824-3256

Contents

List of Tables

1 Introduction

This document describes mathematical and computational considerations pertaining to size distributions. The application of statistical theory to define meaningful and measurable parameters for defining generic size distributions is presented in §2. The remaining sections apply these definitions to the size distributions most commonly used to describe clouds and aerosol size distributions in the meteorological literature. Currently, only the lognormal distribution is presented.

1.1 Nomenclature

nomenclature There is a bewildering variety of nomenclature associated with size distributions, probability density functions, and statistics thereof. The nomenclature in this article generally follows the standard references, [see, e.g., **?????**], at least where those references are in agreement. Quantities whose nomenclature is often confusing, unclear, or simply not standardized are discussed in the text.

1.2 Distribution Function

This section follows the carefully presented discussion of **?**. The *size distribution* function $n_n(r)$ is defined such that $n_n(r)\,\mathrm{d}r$ is the total concentration (number per unit volume of air, or # m^{-3}) of particles with sizes in the domain $[r, r + \mathrm{d}r]$. The total number concentration of particles N_0 is obtained by integrating $n_n(r)$ over all sizes

$$N_0 = \int_0^\infty n_n(r)\,\mathrm{d}r \tag{1}$$

The size distribution function is also called the *spectral density function*. The dimensions of $n_n(r)$ and N_0 are # m^{-3} m^{-1} and # m^{-3}, respectively. Note that $n_n(r)$ is not normalized (unless N_0 happens to equal 1.0).

Often N_0 is not an observable quantity. A variety of functional forms, some of which are overloaded for clarity, describe the number concentrations actually measured by instruments. Typically an instrument has a lower detection limit r_{min} and an upper detection limit r_{max} of particle sizes which it can measure.

$$N(r < r_{\mathrm{max}}) = \int_0^{r_{\mathrm{max}}} n_n(r)\,\mathrm{d}r \tag{2}$$

$$N(r > r_{\mathrm{max}}) = \int_{r_{\mathrm{max}}}^\infty n_n(r)\,\mathrm{d}r \tag{3}$$

$$N(r_{\mathrm{min}}, r_{\mathrm{max}}) = N(r_{\mathrm{min}} < r < r_{\mathrm{max}}) = \int_{r_{\mathrm{min}}}^{r_{\mathrm{max}}} n_n(r)\,\mathrm{d}r \tag{4}$$

Equations (2)–(4) define the *cumulative concentration*, *lower bound concentration*, and *truncated concentration*, respectively. The cumulative concentration is used to define the *median radius* $\tilde{r}_n$. Half the particles are larger and half smaller than $\tilde{r}_n$

$$N(r < \tilde{r}_n) = N(r > \tilde{r}_n) = \frac{N_0}{2} \tag{5}$$

These functions are often used to define $n_n(r)$ via

$$n_n(r) = \frac{\mathrm{d}N(r)}{\mathrm{d}r} \tag{6}$$

1.3 Probability Density Function

Describing size distributions is easier when they are normalized into *probability density functions*, or PDFs. In this context, a PDF is a size distribution function normalized to unity over the domain of interest, i.e., $p(r) = C_n n_n(r)$ where the normalization constant C_n is defined such that

$$\int_0^\infty p(r)\,\mathrm{d}r = 1 \tag{7}$$

In the following sections we usually work with PDFs because this normalization property is very convenient mathematically. Comparing (7) and (1), it is clear that the normalization constant C_n which transforms a size distribution function (1) into a PDF $p(r)$ is N_0^{-1}

$$p(r) = \frac{1}{N_0}\,n_n(r) \tag{8}$$

1.3.1 Choice of Independent Variable

The merits of using radius r, diameter D, or some other dimension L, as the independent variable of a size distribution depend on the application. In radiative transfer applications, r prevails in the literature probably because it is favored in electromagnetic and Mie theory. There is, however, a growing recognition of the importance of aspherical particles in planetary atmospheres. Defining an *equivalent radius* or *equivalent diameter* for these complex shapes is not straighforward (consider, e.g., a bullet rosette ice crystal). Important differences exist among the competing definitions, such as *equivalent area spherical radius*, *equivalent volume spherical radius*, [e.g., **??**].

A direct property of aspherical particles which can often be measured, is its maximum dimension, i.e., the greatest distance between any two surface points of the particle. This maximum dimension, usually called L, has proven to be useful for characterizing size distributions of aspherical particles. For a sphere, L is also the diameter. Analyses of mineral dust sediments in ice core deposits or sediment traps, for example, are usually presented in terms of L. The surface area and volume of ice crystals have been computed in terms of power laws of L [e.g., **??**]. Since models usually lack information regarding the shape of particles [exceptions include **??**], most modelers assume spherical particles, especially for aerosols. Thus, the advantages of using the diameter D as the independent variable in size distribution studies include: D is the dimension often reported in measurements; D is more analogous than r to L.

The remainder of this manuscript assumes spherical particles where r and D are equally useful independent variables. Unless explicitly noted, our convention will be to use D as the independent variable. Thus, it is useful to understand the rules governing conversion of PDFs from D to r and the reverse.

Consider two distinct analytic representations of the same underlying size distribution. The first, $n_n^D(D)$, expresses the differential number concentration per unit diameter. The second, $n_n^r(r)$, expresses the differential number concentration per unit radius. Both $n_n^D(D)$ and $n_n^r(r)$ share the same dimensions, $\#\ \mathrm{m}^{-3}\ \mathrm{m}^{-1}$.

$$D = 2r \tag{9}$$

$$\mathrm{d}D = 2\mathrm{d}r \tag{10}$$

$$n_n^D(D)\,\mathrm{d}D = n_n^r(r)\,\mathrm{d}r \tag{11}$$

$$n_n^D(D) = \frac{1}{2}n_n^r(r) \tag{12}$$

2 Statistics of Size Distributions

2.1 Generic

Consider an arbitrary function $g(x)$ which applies over the domain of the size distribution $p(x)$. For now the exact definition of g is irrelevant, but imagine that $g(x)$ describes the variation of some physically meaningful quantity (e.g., area) with size. The *mean value* of g is the integral of g over the domain of the size distribution, weighted at each point by the concentration of particles

$$\overline{g} = \int_0^\infty g(x)\,p(x)\,\mathrm{d}x \tag{13}$$

Once $p(x)$ is known, it is always possible to compute $\overline{g}$ for any desired quantity g. Typical quantities represented by $g(x)$ are size, $g(x) = x$; area, $g(x) = A(x) \propto x^2$; and volume $g(x) = V(x) \propto x^3$. More complicated statistics represented by $g(x)$ include variance, $g(x) = (x - \bar{x})^2$. The remainder of this section considers some of these examples in more detail.

2.2 Mean Size

The *number mean size* $\bar{x}$ of a size distribution $p(x)$ is defined as

$$\bar{x} = \int_0^\infty p(x)\,x\,\mathrm{d}x \tag{14}$$

Synonyms for number mean size include *mean size*, *average size*, *arithmetic mean size*, and *number-weighted mean size* [**?**]. **?** define $\bar{D}_n \equiv \bar{D}$, a convention we adopt in the following.

2.3 Variance

The *variance* σ_x^2 of a size distribution $p(x)$ is defined in accord with the statistical variance of a continuous mathematical distribution.

$$\sigma_x^2 = \int_0^\infty p(x)(x - \bar{x})^2\,\mathrm{d}x \tag{15}$$

The variance measures the mean squared-deviation of the distribution from its mean value. The units of σ_x^2 are m^2. Because σ_x^2 is a complicated function for the standard aerosol and cloud size distributions, many prefer to work with an alternate definition of variance, called the *effective variance*.

The *effective variance* $\sigma_{x,\mathrm{eff}}^2$ of a size distribution $p(x)$ is defined as the variance about the effective size of the distribution, normalized by x_{eff} [e.g., **?**]

$$\sigma_{x,\mathrm{eff}}^2 = \frac{1}{x_{\mathrm{eff}}^2} \int_0^\infty p(x)(x - x_{\mathrm{eff}})^2\,x^2\,\mathrm{d}x \tag{16}$$

Because of the x_{eff}^{-2} normalization, $\sigma_{x,\mathrm{eff}}^2$ is non-dimensional. In the terminology of **?**, $\sigma_{x,\mathrm{eff}}^2 = v$.

2.4 Standard Deviation

The *standard deviation* σ_x of a size distribution $p(x)$ is simply the square root of the variance,

$$\sigma_x = \sqrt{\sigma_x^2} \tag{17}$$

The units of σ_x are m. For standard aerosol and cloud size distributions, σ_x is an ugly expression. Therefore many authors prefer to work with alternate definitions of standard deviation. Unfortunately, the nomenclature for these alternate definitions has not been standardized.

3 Cloud and Aerosol Size Distributions

3.1 Lognormal Distribution

The *lognormal distribution* is perhaps the most commonly used analytic expression in aerosol studies. Table 3 summarizes the standard lognormal distribution parameters. Note that $\tilde{\sigma_g} \equiv \ln \sigma_g$.

The statistics in Table 3 are easy to misunderstand because of the plethora of subtly different definitions. A common mistake is to assume that patterns which seems to apply to one distribution, e.g., the number distribution $n_n(D)$, apply to distributions of all other moments. For example, the number distribution $n_n(D)$ is the *only* distribution for which the moment mean size (i.e., number mean size $\bar{D}_n$) equals the moment-weighted size (i.e., number-weighted size D_n). Also, the number mean size $\bar{D}_n$ differs from the number median size $\tilde{D}_n$ by a factor of $\mathrm{e}^{\tilde{\sigma_g}^2/2}$. But this factor is not constant and depends on the moment of the distribution. For instance, $\bar{D}_s$ differs from $\tilde{D}_s$ by $\mathrm{e}^{\tilde{\sigma_g}^2}$, while $\bar{D}_s$ differs from $\tilde{D}_s$ by $\mathrm{e}^{3\tilde{\sigma_g}^2/2}$. Thus converting from mean diameter to median diameter is not the same for number as for mass distributions.

Table 1: **Lognormal Distribution Relations**[123]

Symbol	Value	Units	Description	Defining Relationship
N_0	N_0	$\#\ \mathrm{m}^{-3}$	Total number concentration	$N_0 = \int_0^\infty n_n(D)\,\mathrm{d}D$
D_0		$\mathrm{m\ m}^{-3}$	Total diameter	$D_0 = \int_0^\infty D n_n(D)\,\mathrm{d}D$
A_0	$\frac{\pi}{4} N_0 \tilde{D}_n^2 \exp(2\tilde{\sigma}_g^2)$	$\mathrm{m}^2\ \mathrm{m}^{-3}$	Total cross-sectional area	$A_0 = \int_0^\infty \frac{\pi}{4} D^2 n_n(D)\,\mathrm{d}D$
S_0	$\pi N_0 \tilde{D}_n^2 \exp(2\tilde{\sigma}_g^2)$	$\mathrm{m}^2\ \mathrm{m}^{-3}$	Total surface area	$S_0 = \int_0^\infty \pi D^2 n_n(D)\,\mathrm{d}D$
V_0	$\frac{\pi}{6} N_0 \tilde{D}_n^3 \exp(9\tilde{\sigma}_g^2/2)$	$\mathrm{m}^3\ \mathrm{m}^{-3}$	Total volume	$V_0 = \int_0^\infty \frac{\pi}{6} D^3 n_n(D)\,\mathrm{d}D$
M_0	$\frac{\pi}{6} N_0 \rho \tilde{D}_n^3 \exp(9\tilde{\sigma}_g^2/2)$	$\mathrm{kg\ m}^{-3}$	Total mass	$M_0 = \int_0^\infty \frac{\pi}{6} \rho D^3 n_n(D)\,\mathrm{d}D$
$\bar{D}$	$\tilde{D}_n \exp(\tilde{\sigma}_g^2/2)$	$\mathrm{m}\ \#^{-1}$	Mean diameter	$N_0\bar{D} = N_0\bar{D}_n = D_0$
$\bar{A}$	$\frac{\pi}{4} \tilde{D}_n^2 \exp(2\tilde{\sigma}_g^2)$	$\mathrm{m}^2\ \#^{-1}$	Mean cross-sectional area	$N_0\bar{A} = N_0\frac{\pi}{4}\bar{D}_s^2 = A_0$
$\bar{S}$	$\pi \tilde{D}_n^2 \exp(2\tilde{\sigma}_g^2)$	$\mathrm{m}^2\ \#^{-1}$	Mean surface area	$N_0\bar{S} = N_0\pi\bar{D}_s^2 = S_0$
$\bar{V}$	$\frac{\pi}{6} \tilde{D}_n^3 \exp(9\tilde{\sigma}_g^2/2)$	$\mathrm{m}^3\ \#^{-1}$	Mean volume	$N_0\bar{V} = N_0\frac{\pi}{6}\bar{D}_v^3 = V_0$
$\bar{M}$	$\frac{\pi}{6} \rho \tilde{D}_n^3 \exp(9\tilde{\sigma}_g^2/2)$	$\mathrm{kg}\ \#^{-1}$	Mean mass	$N_0\bar{M} = N_0\frac{\pi}{6}\rho\bar{D}_v^3 = M_0$
N_0	$\frac{6}{\pi\rho} M_0 \tilde{D}_n^{-3} \exp(-9\tilde{\sigma}_g^2/2)$	$\#\ \mathrm{m}^{-3}$	Number concentration	
$\tilde{D}_n$	$\left(\frac{6M_0}{\pi N_0 \rho}\right)^{1/3} \exp(-3\tilde{\sigma}_g^2/2)$	m	Median diameter	
$\tilde{D}_n$	$\bar{D}_n \exp(-\tilde{\sigma}_g^2/2)$	m	Median diameter, Scaling diameter, Number median diameter. Half of particles are larger than, and half smaller than, $\tilde{D}_n$	$\int_0^{\tilde{D}_n} n_n(D)\,\mathrm{d}D = \frac{N_0}{2}$

Table 1: (continued)

Symbol	Value	Units	Description	Defining Relationship
$\bar{D}_n$, $\bar{D}$, D_n	$\tilde{D}_n \exp(\tilde{\sigma}_g^2/2)$	m	Mean diameter, Average diameter, Number-weighted mean diameter	$\bar{D}_n = \frac{1}{N_0}\int_0^\infty D n_n(D)\,\mathrm{d}D$
$\bar{D}_s$	$\tilde{D}_n \exp(\tilde{\sigma}_g^2)$	m	Surface mean diameter	$N_0 \pi \bar{D}_s^2 = N_0 \bar{S} = S_0$
$\bar{D}_v$	$\tilde{D}_n \exp(3\tilde{\sigma}_g^2/2)$	m	Volume mean diameter, Mass mean diameter	$N_0 \frac{\pi}{6} \bar{D}_v^3 = N_0 \bar{V} = V_0$
$\tilde{D}_s$	$\tilde{D}_n \exp(2\tilde{\sigma}_g^2)$	m	Surface median diameter	$\int_0^{\tilde{D}_s} \pi D^2 n_n(D)\,\mathrm{d}D = \frac{S_0}{2}$
D_s, D_{eff}	$\tilde{D}_n \exp(5\tilde{\sigma}_g^2/2)$	m	Area-weighted mean diameter, effective diameter	$D_x = \frac{1}{A_0}\int_0^\infty D \frac{\pi}{4} D^2 n_n(D)\,\mathrm{d}D$
$\tilde{D}_v$	$\tilde{D}_n \exp(3\tilde{\sigma}_g^2)$	m	Volume median diameter Mass median diameter	$\int_0^{\tilde{D}_v} \frac{\pi}{6} D^3 n_n(D)\,\mathrm{d}D = \frac{V_0}{2}$
D_v	$\tilde{D}_n \exp(7\tilde{\sigma}_g^2/2)$	m	Mass-weighted mean diameter, Volume-weighted mean diameter	$D_v = \frac{1}{V_0}\int_0^\infty D \frac{\pi}{6} D^3 n_n(D)\,\mathrm{d}D$

Table 2 lists applies the relations in Table 3 to specific size distributions typical of tropospheric aerosols. **?** and **?** describe measurements and transport of dust across the Atlantic and Pacific, respectively. **?** summarize historical measurements of dust size distributions, and analyze the influence of measurement technique on the derived size distribution. They show the derived size distribution is strongly sensitive to the measurment technique. During PRIDE, measured $\tilde{D}_v$ varied from 2.5–9 μm depending on the instrument employed. **?** show that the change in mineral dust size distribution across the sub-tropical Atlantic is consistent with a slight updraft of ~ 0.33 cm s^{-1} during transport. **?** and **?** show that the effects of asphericity on particle settling velocity play an important role in maintaining the large particle tail of the size distribution during long range transport.

Table 3 applies the relations in Table 3 to specific size distributions typical of tropospheric aerosols.

3.1.1 Distribution Function

The *lognormal distribution function* is

$$n_n(D) = \frac{1}{\sqrt{2\pi}\, D \ln \sigma_g} \exp\left[-\frac{1}{2}\left(\frac{\ln(D/\tilde{D}_n)}{\ln \sigma_g}\right)^2\right] \tag{18}$$

One of the most confusing aspects of size distributions in the meteorological literature is in the usage of σ_g, which is frequently called the *geometric standard deviation*. Some researchers [e.g., **?**] denote by σ_g what most denote by $\ln \sigma_g$. Thus the form of the lognormal distribution function *sometimes* appears

$$n_n(D) = \frac{1}{\sqrt{2\pi}\, \sigma_g D} \exp\left[-\frac{1}{2}\left(\frac{\ln(D/\tilde{D}_n)}{\sigma_g}\right)^2\right] \tag{19}$$

In practice, (18) is used more widely than (19) but the definition of σ_g in the latter may be more satisfactory from a mathematical point of view [**?**] (and it subsumes the "ln", which reduces typing). We adopt (18) in the following, and sometimes simplify formulae by using a convenient definition of $\tilde{\sigma_g} \equiv \ln \sigma_g$. One is occasionally given a "standard deviation" or "geometric standard deviation" parameter without clear specification whether it represents σ_g (or $\ln \sigma_g$, or $\exp \sigma_g$, or σ_x) in (17), (18), or (19). A useful rule of thumb is that σ_g in (18) and e^{σ_g} in (19) are usually near 2.0 for realistic aerosol populations. Since we adopted (18), physically realistic values of σ_g presented in this manuscript will be near 2.0.

Note that direct substitution of $D = 2r$ into (18) yields

$$\begin{aligned} n_n(D) &= \frac{1}{\sqrt{2\pi}\, 2r \ln \sigma_g} \exp\left[-\frac{1}{2}\left(\frac{\ln(2r/2\tilde{r}_n)}{\ln \sigma_g}\right)^2\right] \\ &= \frac{1}{2}\frac{1}{\sqrt{2\pi}\, r \ln \sigma_g} \exp\left[-\frac{1}{2}\left(\frac{\ln(r/\tilde{r}_n)}{\ln \sigma_g}\right)^2\right] \\ &= \frac{1}{2} n_n^r(r) \end{aligned} \tag{20}$$

in agreement with (12).

Table 2: **Lognormal Size Distribution Statistics**

$\tilde{D}_n$ μm	$\tilde{D}_v$ μm	σ_g	M	Ref.[a]
?[b]				
0.003291	0.0111	1.89	2.6×10^{-4}	2
0.5972	2.524	2.0[c]	0.781	2, 4
7.575	42.1	2.13	0.219	2
?[d]				
0.1600	0.832	2.1	0.036	3
1.401	4.82	1.90	0.957	3
9.989	19.38	1.60	0.007	3
? [e]				
0.08169	0.27	1.88		1
0.8674	5.6	2.2		1
28.65	57.6	1.62		1

[a]*References:* 1, **?**; 2, **?**; 3, **?**; 4, **?**. Values reported in literature were converted to values shown in table using the analytic expressions summarized in Table 3. Usually this entailed deriving $\tilde{D}_n$ given $\tilde{D}_v$, $\tilde{r}_v$, or $\tilde{D}_s$.

[b]Background Desert Model from **?**, p. 75 Table 1.

[c]$\sigma_g = 2.0$ for transport mode follows **?**, p. 10581, Table 1.

[d]**?**, p. 73 Table 2. These are the "background" modes of D'Almeida (1987).

[e]Detailed fits to dust sampled over Colorado and Texas in **?**, p. 2080 Table 1. Original values have been converted from radius to diameter. M was not given. **?** showed soil aerosol could be represented with three modes which they dubbed, in order of increasing size, modes C, A, and B. Mode A is the mineral dust transport mode, seen in source regions and downwind. Mode B is seen in the source soil itself, and in the atmosphere during dust events. Mode C is seen most everywhere, but does not usually correlate with local dust amount. Mode C is usually a global, aged, background, anthropogenic aerosol, typically rich in sulfate and black carbon. Sometimes, however, Mode C has a mineral dust component. Modes C and B are averages from **?** Table 1 p. 2080. Mode B is based on the summary recommendation that $\tilde{r}_s = 1.5$ and $\sigma_g = 2.2$.

Table 3: **Analytic Lognormal Size Distribution Statistics** [a][b]

$\tilde{D}_n$ µm	D_n µm	$\tilde{D}_s$ µm	D_s µm	$\tilde{D}_v$ µm	D_v µm	σ_g
0.1861	0.2366	0.4864	0.6184	0.7863	1.0	2.0
0.2366	0.3008	0.6184	0.7863	1.0	1.271	2.0
0.3009	0.3825	0.7864	1.0	1.271	1.616	2.0
0.3825	0.4864	1.0	1.271	1.616	2.055	2.0
0.5915	0.7521	1.546	1.966	2.5	3.178	2.0
0.7864	1.0	2.056	2.614	3.323	4.225	2.0
1.0	1.272	2.614	3.323	4.227	5.373	2.0
1.183	1.504	3.092	3.932	5.0	6.356	2.0
2.366	3.008	6.184	7.863	10.0	12.71	2.0

[a] Shown are statistics for each moment equalling 1 µm, and for $\tilde{D}_v = 2.5, 5.0, 10.0$ µm.
[b] $\tilde{D}_n$, $\tilde{D}_s$, and $\tilde{D}_v$ are number, surface, and volume median diameters, respectively. D_n, D_s, and D_v are number, surface, and volume-weighted diameters.

3.1.2 Variance

According to (15), the variance σ_D^2 of the lognormal distribution (18) is

$$\sigma_D^2 = \frac{1}{\sqrt{2\pi}\ln\sigma_g}\int_0^\infty \frac{1}{D}\exp\left[-\frac{1}{2}\left(\frac{\ln(D/\tilde{D}_n)}{\ln\sigma_g}\right)^2\right](D-\bar{D})^2\,\mathrm{d}D \tag{21}$$

3.1.3 Bounded Distribution

The statistical properties of a bounded lognormal distribution are expressed in terms of the error function (§5.2). The cumulative concentration bounded by $D_{\max}$ is given by applying (2) to (18)

$$N(D < D_{\max}) = \frac{N_0}{\sqrt{2\pi}\,\ln\sigma_g}\int_0^{D_{\max}} \frac{1}{D}\exp\left[-\frac{1}{2}\left(\frac{\ln(D/\tilde{D}_n)}{\ln\sigma_g}\right)^2\right]\mathrm{d}D \tag{22}$$

We make the change of variable $z = (\ln D - \ln\tilde{D}_n)/\sqrt{2}\,\ln\sigma_g$

$$\begin{aligned}
z &= (\ln D - \ln\tilde{D}_n)/\sqrt{2}\,\ln\sigma_g \\
D &= \tilde{D}_n \mathrm{e}^{\sqrt{2}\,z\ln\sigma_g} \\
&= \tilde{D}_n\sigma_g^{\sqrt{2}\,z} \\
\mathrm{d}z &= (\sqrt{2}\,D\ln\sigma_g)^{-1}\,\mathrm{d}D \\
\mathrm{d}D &= \sqrt{2}\,\ln\sigma_g\tilde{D}_n\mathrm{e}^{\sqrt{2}\,z\ln\sigma_g}\,\mathrm{d}z \\
&= \sqrt{2}\,\ln\sigma_g\tilde{D}_n\sigma_g^{\sqrt{2}\,z}\,\mathrm{d}z
\end{aligned} \tag{23}$$

which maps $D \in (0, D_{\max})$ into $z \in (-\infty, \ln D_{\max} - \ln \tilde{D}_n)/\sqrt{2}\ln\sigma_g)$. In terms of z we obtain

$$\begin{aligned}
N(D < D_{\max}) &= \frac{N_0}{\sqrt{2\pi}\,\ln\sigma_g}\int_{-\infty}^{(\ln D_{\max}-\ln\tilde{D}_n)/\sqrt{2}\,\ln\sigma_g} \frac{1}{\tilde{D}_n \mathrm{e}^{\sqrt{2}\,z\ln\sigma_g}}\mathrm{e}^{-z^2}\sqrt{2}\,\ln\sigma_g\tilde{D}_n\mathrm{e}^{\sqrt{2}\,z\ln\sigma_g}\,\mathrm{d}z \\
&= \frac{N_0}{\sqrt{\pi}}\int_{-\infty}^{(\ln D_{\max}-\ln\tilde{D}_n)/\sqrt{2}\,\ln\sigma_g}\mathrm{e}^{-z^2}\,\mathrm{d}z \\
&= \frac{N_0}{\sqrt{\pi}}\left(\int_{-\infty}^{0}\mathrm{e}^{-z^2}\,\mathrm{d}z + \int_{0}^{(\ln D_{\max}-\ln\tilde{D}_n)/\sqrt{2}\,\ln\sigma_g}\mathrm{e}^{-z^2}\,\mathrm{d}z\right) \\
&= \frac{N_0}{2}\left(\frac{2}{\sqrt{\pi}}\int_{0}^{+\infty}\mathrm{e}^{-z^2}\,\mathrm{d}z + \frac{2}{\sqrt{\pi}}\int_{0}^{(\ln D_{\max}-\ln\tilde{D}_n)/\sqrt{2}\,\ln\sigma_g}\mathrm{e}^{-z^2}\,\mathrm{d}z\right) \\
&= \frac{N_0}{2}\left[\mathrm{erf}(\infty) + \mathrm{erf}\left(\frac{\ln(D_{\max}/\tilde{D}_n)}{\sqrt{2}\,\ln\sigma_g}\right)\right] \\
&= \frac{N_0}{2} + \frac{N_0}{2}\,\mathrm{erf}\left(\frac{\ln(D_{\max}/\tilde{D}_n)}{\sqrt{2}\,\ln\sigma_g}\right)
\end{aligned} \tag{24}$$

where we have used the properties of the error function (§5.2). The same procedure can be performed to compute the cumulative concentration of particles smaller than $D_{\min}$. When $N(D < D_{\min})$ is subtracted from (24) we obtain the truncated concentration (4)

$$N(D_{\min}, D_{\max}) = \frac{N_0}{2}\left[\mathrm{erf}\left(\frac{\ln(D_{\max}/\tilde{D}_n)}{\sqrt{2}\,\ln\sigma_g}\right) - \mathrm{erf}\left(\frac{\ln(D_{\min}/\tilde{D}_n)}{\sqrt{2}\,\ln\sigma_g}\right)\right] \tag{25}$$

We are also interested in the bounded mass distribution, i.e., the mass of particles lying between $D_{\min}$ and $D_{\max}$. The mass distribution is related to the number distribution by

$$n_v(D) = \frac{\pi}{6}\rho D^3 n_n(D) \tag{26}$$

so that we simply let $\tilde{D}_n = \tilde{D}_v$ in (25) and we obtain

$$V(D_{\min}, D_{\max}) = \frac{N_0}{2}\left[\mathrm{erf}\left(\frac{\ln(D_{\max}/\tilde{D}_v)}{\sqrt{2}\,\ln\sigma_g}\right) - \mathrm{erf}\left(\frac{\ln(D_{\min}/\tilde{D}_v)}{\sqrt{2}\,\ln\sigma_g}\right)\right] \tag{27}$$

3.1.4 Statistics of Bounded Distributions

All of the relationships given in Table 3 may be re-expressed in terms of truncated lognormal distributions, but doing so is tedious, and requires new terminology. Instead we derive the expression for a typical size distribution statistic, and allow the reader to generalize. We generalize (13) to consider

$$\overline{g}^* = \int_{D_{\min}}^{D_{\max}} D\,p^*(D)\,\mathrm{d}D \tag{28}$$

Note the domain of integration, $D \in (D_{\min}, D_{\max})$, reflects the fact that we are considering a bounded distribution. The superscript * indicates that the average statistic refers to a truncated distribution and reminds us that $\overline{g}^* \neq \overline{g}$. Defining a closed form expression for $p^*(D)$ requires some consideration. This truncated distribution has N_0^* defined by (25), and is completely specified on $D \in (0, \infty)$ by

$$p^*(D) = \begin{cases} 0 & , \quad 0 < D < D_{\min} \\ N(D_{\min}, D_{\max})\, p(D)/N_0 & , \quad D_{\min} \leq D \leq D_{\max} \\ 0 & , \quad D_{\max} < D < \infty \end{cases} \tag{29}$$

The difficulty is that the three parameters of the lognormal distribution, $\tilde{D}_n$, σ_g, and N_0 are defined in terms of an untruncated distribution. Using (25) we can write

$$p^*(D) = \frac{1}{N_0^*}\, n_n(D) N_0^* = N(D_{\min}, D_{\max}) \tag{30}$$

If we think of p^* order to be properly normalized to unity, note that (fxm) Thus when we speak of truncated distributions it is important to keep in mind that the parameters $\tilde{D}_n$, σ_g, and N_0 refer to the untruncated distribution.

The properties of the truncated distribution will be expressed in terms of $\tilde{D}_n^*$, σ_g^*, and N_0^*, respectively.

Consider the mean size, D. In terms of (13) we have $g(D) = D$ so that

$$\bar{D} = \int_{D_{\min}}^{D_{\max}} D\, p(D)\, \mathrm{d}x \tag{31}$$

3.1.5 Overlapping Distributions

Consider the problem of distributing I independent and possibly *overlapping distributions* of particles into J independent and possibly overlapping distributions of particles. To reify the problem we call the I bins the source bins (these bins represent the parent size distributions in mineral dust source areas) and the J bins as sink bins (which represent sizes transported in the atmosphere). Typically we know the total mass M_0 or number N_0 of source particles to distribute into the sink bins and we know the fraction of the total mass to distribute which resides in each source distribution, M_i. The problem is to determine matrices of overlap factors $N_{i,j}$ and $M_{i,j}$ which determine what number and mass fraction, respectively, of each source bin i is blown into each sink bin j.

The mass and number fractions contained by the source distributions are normalized such that

$$\sum_{i=1}^{I} M_i = \sum_{i=1}^{I} N_i = 1 \tag{32}$$

In the case of dust emissions, M_i and N_i may vary with spatial location.

The overlap factors $N_{i,j}$ and $M_{i,j}$ are defined by the relations

$$\begin{aligned} N_j &= \sum_{i=1}^{I} N_{i,j} N_i \\ &= N_0 \sum_{i=1}^{I} N_{i,j} N_i \end{aligned} \tag{33}$$

$$\begin{aligned} M_j &= \sum_{i=1}^{I} M_{i,j} M_i \\ &= M_0 \sum_{i=1}^{I} M_{i,j} M_i \end{aligned} \tag{34}$$

Using (25) and (32) we find

$$N_{i,j} = \frac{1}{2}\left[\operatorname{erf}\left(\frac{\ln(D_{\max,j}/\tilde{D}_{n,i})}{\sqrt{2}\ln\sigma_{g,i}}\right) - \operatorname{erf}\left(\frac{\ln(D_{\min,j}/\tilde{D}_{n,i})}{\sqrt{2}\ln\sigma_{g,i}}\right)\right] \tag{35}$$

$$M_{i,j} = \frac{1}{2}\left[\operatorname{erf}\left(\frac{\ln(D_{\max,j}/\tilde{D}_{v,i})}{\sqrt{2}\ln\sigma_{g,i}}\right) - \operatorname{erf}\left(\frac{\ln(D_{\min,j}/\tilde{D}_{v,i})}{\sqrt{2}\ln\sigma_{g,i}}\right)\right] \tag{36}$$

fxm: The mathematical derivation appears correct but the overlap factor appears to asymptote to 0.5 rather than to 1.0 for $D_{\max} \gg \tilde{D}_n \gg D_{\min}$.

A mass distribution has the same form as a lognormal number distribution but has a different median diameter. Thus the overlap matrix elements apply equally to mass and number distributions depending on the median diameter used in the following formulae. For the case where both source and sink distributions are complete lognormal distributions,

$$M(D) = \sum_{i=1}^{i=I} M_i(D)$$

3.1.6 Median Diameter

Substituting $D = \tilde{D}_n$ into (24) we obtain

$$N(D < \tilde{D}_n) = \frac{N_0}{2} \tag{37}$$

Thus the validity of $\tilde{D}_n$ as the median diameter is now proven (5). The lognormal distribution is the only distribution known (to us) which is most naturally expressed in terms of its median diameter.

3.1.7 Multimodal Distributions

Realistic particle size distributions may be expressed as an appropriately weighted sum of individual modes.

$$n_n(D) = \sum_{i=1}^{I} n_n^i(D) \tag{38}$$

where $n_n^i(D)$ is the number distribution of the ith component mode[4]. Such particle size distributions are called *multimodal istributions* because they contain one maximum for each component distribution. Generalizing (1), the total number concentration becomes

$$
\begin{aligned}
N_0 &= \sum_{i=1}^{I} \int_0^\infty n_n^i(D)\,\mathrm{d}D \\
&= \sum_{i=1}^{I} N_0^i
\end{aligned}
\tag{39}
$$

where N_0^i is the total number concentration of the ith component mode.

The median diameter of a multimodal distribution is obtained by following the logic of (22)–(25). The number of particles smaller than a given size is

$$
N(D < D_{\max}) = \sum_{i=1}^{I} \frac{N_0^i}{2} + \frac{N_0^i}{2}\,\mathrm{erf}\left(\frac{\ln(D_{\max}/\tilde{D}_n^i)}{\sqrt{2}\,\ln\sigma_g^i}\right) \tag{40}
$$

$$
\tag{41}
$$

For the median particle size, $D_{\max} \equiv \tilde{D}_n$, and we can move the unknown $\tilde{D}_n$ to the LHS yielding

$$
\begin{aligned}
\sum_{i=1}^{I} \frac{N_0^i}{2} + \frac{N_0^i}{2}\,\mathrm{erf}\left(\frac{\ln(\tilde{D}_n/\tilde{D}_n^i)}{\sqrt{2}\,\ln\sigma_g^i}\right) &= \frac{N_0}{2} \\
\sum_{i=1}^{I} N_0^i\,\mathrm{erf}\left(\frac{\ln(\tilde{D}_n/\tilde{D}_n^i)}{\sqrt{2}\,\ln\sigma_g^i}\right) &= 0
\end{aligned}
\tag{42}
$$

where we have used $N_0 = \sum_i^I N_0^i$. Obtaining $\tilde{D}_n$ for a multimodal distribution requires numerically solving (42) given the N_0^i, $\tilde{D}_n^i$, and σ_g^i.

3.2 Higher Moments

It is often useful to compute higher moments of the number distribution. Each factor of the independent variable weighting the number distribution function $n_n(D)$ in the integrand of (14) counts as a *moment*. The kth moment of $n_n(D)$ is

$$
F(k) = \int_0^\infty n_n(D) D^k\,\mathrm{d}D \tag{43}
$$

The statistical properties of higher moments of the lognormal size distribution may be obtained by direct integration of (43).

$$
\begin{aligned}
F(k) &= \frac{N_0}{\sqrt{2\pi}\,\ln\sigma_g} \int_0^\infty \frac{1}{D} \exp\left[-\frac{1}{2}\left(\frac{\ln(D/\tilde{D}_n)}{\ln\sigma_g}\right)^2\right] D^k\,\mathrm{d}D \\
&= \frac{N_0}{\sqrt{2\pi}\,\ln\sigma_g} \int_0^\infty D^{k-1} \exp\left[-\frac{1}{2}\left(\frac{\ln(D/\tilde{D}_n)}{\ln\sigma_g}\right)^2\right] \mathrm{d}D
\end{aligned}
\tag{44}
$$

[4]Throughout this section the i superscript represents an index of the component mode, not an exponent.

We make the same change of variable $z = (\ln D - \ln \tilde{D}_n)/\sqrt{2}\,\ln\sigma_g$ as in (23). This maps $D \in (0, +\infty)$ into $z \in (-\infty, +\infty)$. In terms of z we obtain

$$
\begin{aligned}
F(k) &= \frac{N_0}{\sqrt{2\pi}\,\ln\sigma_g}\int_{-\infty}^{+\infty}(\tilde{D}_n \mathrm{e}^{\sqrt{2}\,z\ln\sigma_g})^{k-1}\mathrm{e}^{-z^2}\sqrt{2}\,\ln\sigma_g \tilde{D}_n \mathrm{e}^{\sqrt{2}\,z\ln\sigma_g}\,\mathrm{d}z \\
&= \frac{N_0}{\sqrt{\pi}}\int_{-\infty}^{+\infty}(\tilde{D}_n \mathrm{e}^{\sqrt{2}\,z\ln\sigma_g})^{k}\mathrm{e}^{-z^2}\,\mathrm{d}z \\
&= \frac{N_0\tilde{D}_n^k}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\mathrm{e}^{\sqrt{2}kz\ln\sigma_g}\mathrm{e}^{-z^2}\,\mathrm{d}z \\
&= \frac{N_0\tilde{D}_n^k}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\mathrm{e}^{-z^2+\sqrt{2}kz\ln\sigma_g}\,\mathrm{d}z \\
&= \frac{N_0\tilde{D}_n^k}{\sqrt{\pi}}\sqrt{\pi}\exp\left(\frac{2k^2\ln^2\sigma_g}{4}\right) \\
&= N_0\tilde{D}_n^k\exp(\tfrac{1}{2}k^2\ln^2\sigma_g) \qquad (45)
\end{aligned}
$$

where we have used (54) with $\alpha = 1$ and $\beta = \sqrt{2}k\ln\sigma_g$.

Applying the formula (45) to the first five moments of the lognormal distribution function we obtain

$$
\begin{aligned}
F(0) &= \int_0^\infty n_n(D)\,\mathrm{d}D &&= N_0 &&= N_0 &&= N_0 \\
F(1) &= \int_0^\infty n_n(D)D\,\mathrm{d}D &&= N_0\tilde{D}_n\exp(\tfrac{1}{2}\ln^2\sigma_g) &&= D_0 &&= N_0\bar{D}_n \\
F(2) &= \int_0^\infty n_n(D)D^2\,\mathrm{d}D &&= N_0\tilde{D}_n^2\exp(2\ln^2\sigma_g) &&= \frac{S_0}{\pi} &&= N_0\bar{D}_s^2 \qquad (46) \\
F(3) &= \int_0^\infty n_n(D)D^3\,\mathrm{d}D &&= N_0\tilde{D}_n^3\exp(\tfrac{9}{2}\ln^2\sigma_g) &&= \frac{6V_0}{\pi} &&= N_0\bar{D}_v^3 \\
F(4) &= \int_0^\infty n_n(D)D^4\,\mathrm{d}D &&= N_0\tilde{D}_n^4\exp(8\ln^2\sigma_g)
\end{aligned}
$$

The first few moments of the number distribution are related to measurable properties of the size distribution. In particular, $F(k = 0)$ is the number concentration. Other quantities of merit are ratios of consecutive moments. For example, the volume-weighted diameter D_v is computed by weighted each diameter by the volume of particles at that diameter and then normalizing by the total volume of all particles.

$$
\begin{aligned}
D_v &= \int_0^\infty D\frac{\pi}{6}D^3 n_n(D)\,\mathrm{d}D \Big/ \int_0^\infty \frac{\pi}{6}D^3 n_n(D)\,\mathrm{d}D \\
&= \int_0^\infty D^4 n_n(D)\,\mathrm{d}D \Big/ \int_0^\infty D^3 n_n(D)\,\mathrm{d}D \\
&= F(4)/F(3) \\
&= \frac{N_0\tilde{D}_n^4\exp(8\ln^2\sigma_g)}{N_0\tilde{D}_n^3\exp(\frac{9}{2}\ln^2\sigma_g)} \\
&= \tilde{D}_n\exp(\tfrac{7}{2}\ln^2\sigma_g) \qquad (47)
\end{aligned}
$$

The *surface-weighted diameter* D_s is defined analogously to D_v. Following (47) it is easy to show that

$$
\begin{aligned}
D_s &= F(3)/F(2) \\
&= \frac{N_0 \tilde{D}_n^3 \exp(\frac{9}{2}\ln^2\sigma_g)}{N_0 \tilde{D}_n^2 \exp(2\ln^2\sigma_g)} \\
&= \tilde{D}_n \exp(\tfrac{5}{2}\ln^2\sigma_g)
\end{aligned}
\tag{48}
$$

Moment-weighted diameters, such as the volume-weighted diamter D_v 47, are useful in predicting behavior of disperse distributions. A disperse mass distribution $n_m(D)$ behaves most like a monodisperse distribution with all mass residing at $D = D_v$. Due to approximations, physical operators may be constrained to act on a single, representative diameter rather than an entire distribution. The "least-wrong" diameter to pick is the moment-weighted diameter most relevant to the process being modeled. For example, D_v best represents the gravitational sedimentation of a distribution of particles. On the other hand, D_s (48) best represents the scattering cross-section of a distribution of particles.

3.2.1 Normalization

We show that (18) is normalized by considering

$$
n_n(D) = \frac{C_n}{D}\exp\left[-\frac{1}{2}\left(\frac{\ln(D/\tilde{D}_n)}{\ln\sigma_g}\right)^2\right]
\tag{49}
$$

where C_n is the normalization constant determined by (7). First we change variables to $y = \ln(D/\tilde{D}_n)$

$$
\begin{aligned}
y &= \ln D - \ln\tilde{D}_n \\
D &= \tilde{D}_n e^y \\
dy &= D^{-1}\,dD \\
dD &= \tilde{D}_n e^y\,dy
\end{aligned}
\tag{50}
$$

This transformation maps $D \in (0, +\infty)$ into $y \in (-\infty, +\infty)$. In terms of y, the normalization condition (7) becomes

$$
\begin{aligned}
\int_{-\infty}^{+\infty} \frac{C_n}{\tilde{D}_n \exp y}\exp\left[-\frac{1}{2}\left(\frac{y}{\ln\sigma_g}\right)^2\right]\tilde{D}_n \exp^y dy &= 1 \\
\int_{-\infty}^{+\infty} C_n \exp\left[-\frac{1}{2}\left(\frac{y}{\ln\sigma_g}\right)^2\right] dy &= 1
\end{aligned}
$$

Next we change variables to $z = y/\ln\sigma_g$

$$
\begin{aligned}
z &= y/\ln\sigma_g \\
y &= z\ln\sigma_g \\
dz &= (\ln\sigma_g)^{-1}\,dy \\
dy &= \ln\sigma_g\,dz
\end{aligned}
\tag{51}
$$

Soil Texture	$\tilde{D}_n$	σ_g	Description
Sand			Sand
Silt			Silt
Clay			Clay
Soil Texture	$\tilde{D}_n$	σ_g	Description
Sand			Sand
Silt			Silt
Clay			Clay

Table 4: Source size distribution associated with surface soil texture data of **?** and of **?**.

This transformation does not change the limits of integration and we obtain

$$\begin{aligned} \int_{-\infty}^{+\infty} C_n \exp\left(\frac{-z^2}{2}\right) \ln\sigma_g \, \mathrm{d}z &= 1 \\ C_n \sqrt{2\pi} \ln\sigma_g &= 1 \\ C_n &= \frac{1}{\sqrt{2\pi}\ln\sigma_g} \end{aligned} \tag{52}$$

In the above we used the well-known normalization property of the Gaussian distribution function, $\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2/2}\,\mathrm{d}x = \sqrt{2\pi}$ (53).

4 Implementation in NCAR models

The discussion thus far has centered on the theoretical considerations of size distributions. In practice, these ideas must be implemented in computer codes which model, e.g., Mie scattering parameters or thermodynamic growth of aerosol populations. This section describes how these ideas have been implemented in the NCAR-Dust and Mie models.

4.1 NCAR-Dust Model

The NCAR-Dust model uses as input a time invariant dataset of surface soil size distribution. The two such datasets currently used are from **?** and from IBIS [**?**]. The **?** dataset provides global information for three soil texture types: sand, clay and silt. At each gridpoint, the mass flux of dust is partitioned into mass contributions from each of these soil types. To accomplish this, the partitioning scheme assumes a size distribution for the source soil of the deflated particles. Table 4 lists the lognormal distribution parameters associated with the surface soil texture data of **?** and of **?**. The dust model is a size resolving aerosol model. Thus, overlap factors are computed to determine the fraction of each parent size type which is mobilized into each atmospheric dust size bin during a deflation event.

4.2 Mie Scattering Model

This section documents the use of the Mie scattering code `mie`.

4.2.1 Input switches

Compute size distribution characteristics of a lognormal distribution

```
mie -dbg -no_mie --psd\_typ=lognormal --sz_grd=log --sz_mnm=0.01 \
--sz_mxm=10.0 --sz_nbr=300 --rds_nma=0.4 --gsd_anl=2.2
mie -dbg -no_mie --psd\_typ=lognormal --sz_grd=log --sz_mnm=1.0 \
--sz_mxm=10.0 --sz_nbr=25 --rds_nma=2.0 --gsd_anl=2.2
```

Table 13 summarizes the *command line arguments* available to characterize aerosol distributions in the `mie` program.

Table 5: **Command Line Switches for `mie` code**[56]

Switch	Purpose	Default	Units
	Boolean flags		
`--abc_flg`	Alphabetize output with ncks	true	Flag
`--coat_flg`	Assume coated spheres	false	Flag
`--drv_rds_nma_flg`	Derive rds_nma from bin boundaries	false	Flag
`--fdg_flg`	Tune the extinction of a particular band	false	Flag
`--hrz_flg`	Print size-resolved optical properties at debug wavelength	false	Flag
`--idx_rfr_mdm_usr_flg`	Refractive index of medium is user-specified	false	Flag
`--idx_rfr_mntl_usr_flg`	Mantle refractive index is user-specified	false	Flag
`--idx_rfr_prt_usr_flg`	Refractive index of particle is user-specified	false	Flag
`--mca_flg`	Multi-component aerosol with effective medium approximation	false	Flag
`--mie_flg`	Perform mie scattering calculation	true	Flag
`--ss_alb_flg`	Manually set single scattering albedo	false	Flag
`--tst_flg`	Perform self-test	false	Flag
`--wrn_ntp_flg`	Print WARNINGs from `ntp_vec()`	true	Flag
	Variables		
`--RH_lqd`	Relative humidity w/r/t liquid water	0.8	Fraction
`--aer_sng`	Aerosol type	"dust_like"	String
`--asp_rat_lps_dfl`	Ellipsoidal aspect ratio	1.0	Fraction
`--bnd_SW_LW`	Boundary between SW and LW weighting	5.0×10^{-6}	m
`--bnd_nbr`	Number of sub-bands per output band	1	Number
`--cnc_nbr_anl_dfl`	Number concentration analytic, default	1.0	$\# \ m^{-3}$
`--cnc_nbr_pcp_anl`	Number concentration analytic, raindrop	1.0	$\# \ m^{-3}$

Table 5: (continued)

Switch	Purpose	Default	Units
`--cpv_foo`	Intrinsic computational precision temporary variable	0.0	Fraction
`--dmn_nbr_max`	Maximum number of dimensions allowed in single variable in output file	2	Number
`--dmt_dtc`	Diameter of detector	0.001	m
`--dmt_nma_mcr`	Number median analytic diameter	`cmd_ln_dfl`	μm
`--dmt_pcp_nma_mcr`	Diameter number median analytic, raindrop, microns	1000.0	μm
`--dmt_swa_mcr`	Surface area weighted mean diameter analytic	`cmd_ln_dfl`	μm
`--dmt_vma_mcr`	Volume median diameter analytic	`cmd_ln_dfl`	μm
`--dns_aer`	Aerosol density	0.0	kg m^{-3}
`--dns_mdm`	Density of medium	0.0	kg m^{-3}
`--doy`	Day of year [1.0..366.0)	135.0	day
`--drc_dat`	Data directory	`/data/zender/aca`	String
`--drc_in`	Input directory	`${HOME}/nco/data`	String
`--drc_out`	Output directory	`${HOME}/c++`	String
`--dsd_dbg_mcr`	Debugging size for raindrops	1000.0	μm
`--dsd_mnm_mcr`	Minimum diameter in raindrop distribution	999.0	μm
`--dsd_mxm_mcr`	Maximum diameter in raindrop distribution	1001.0	μm
`--dsd_nbr`	Number of raindrop size bins	1	Number
`--dst_lbl`	Label for FORTRAN block data	“foo”	String
`--fdg_idx`	Band to tune by fdg_val	0	Index
`--fdg_val`	Tuning factor for all bands	1.0	Fraction
`--fl_err`	File for error messages	“cerr”	String

Table 5: (continued)

Switch	Purpose	Default	Units
`--fl_idx_rfr_mdm`	File or function for refractive indices of medium	“”	String
`--fl_idx_rfr_prt`	File or function for refractive indices of particle	“”	String
`--fl_slr_spc`	File or function for solar spectrum	“”	String
`--flt_foo`	Intrinsic float temporary variable	0.0	Fraction
`--flx_LW_dwn_sfc`	Longwave downwelling flux at surface	0.0	W m^{-2}
`--flx_SW_net_gnd`	Solar flux absorbed by ground	450.0	W m^{-2}
`--flx_SW_net_vgt`	Solar flux absorbed by vegetation	0.0	W m^{-2}
`--flx_vlm_pcp_rsl`	Precipitation volume flux, resolved	-1.0	m^{3} m^{-2} s^{-1}
`--gsd_anl_dfl`	Geometric standard deviation, default	2.0	Fraction
`--gsd_pcp_anl`	Geometric standard deviation, raindrop	1.86	Fraction
`--hgt_mdp`	Midlayer height above surface	95.0	m
`--hgt_rfr`	Reference height (i.e., 10 m) at which surface winds are evaluated for dust mobilization	10.0	m
`--hgt_zpd_dps_cmd_ln`	Zero plane displacement height	`cmd_ln_dfl`	m
`--hgt_zpd_mbl`	Zero plane displacement height for erodible surfaces	0.0	m
`--idx_rfr_mdm_img_usr`	Imaginary refractive index of medium	0.0	Fraction
`--idx_rfr_mdm_rl_usr`	Real refractive index of medium	1.0	Fraction
`--idx_rfr_mntl_img_usr`	Imaginary refractive index of mantle	0.0	Fraction
`--idx_rfr_mntl_rl_usr`	Real refractive index of mantle	1.33	Fraction
`--idx_rfr_prt_img_usr`	Imaginary refractive index of particle	0.0	Fraction
`--idx_rfr_prt_rl_usr`	Real refractive index of particle	1.33	Fraction
`--lat_dgr`	Latitude	40.0	°

Table 5: (continued)

Switch	Purpose	Default	Units
`--lbl_sng`	Line-by-line test	"CO2"	String
`--lgn_nbr`	Number of terms in Legendre expansion of phase function	8	Number
`--lnd_frc_dry`	Dry land fraction	1.0	Fraction
`--mdm_sng`	Medium type	"air"	String
`--mmw_aer`	Mean molecular weight	0.0	kg mol^{-1}
`--mno_lng_dps_cmd_ln`	Monin-Obukhov length	`cmd_ln_dfl`	m
`--mss_frc_cly`	Mass fraction clay	0.19	Fraction
`--mss_frc_snd`	Mass fraction sand	0.777	Fraction
`--ngl_nbr`	Number of angles in Mie computation	11	Number
`--oro`	Orography: ocean=0.0, land=1.0, sea ice=2.0	1.0	Fraction
`--pnt_typ_idx`	Plant type index	14	Index
`--prs_mdp`	Environmental pressure	100825.0	Pa
`--prs_ntf`	Environmental surface pressure	`prs_STP`	Pa
`--psd_typ`	Particle size distribution type	"lognormal"	String
`--q_H2O_vpr`	Specific humidity	`cmd_ln_dfl`	kg kg^{-1}
`--rds_ffc_gmm_mcr`	Effective radius of Gamma distribution	50.0	μm
`--rds_nma_mcr`	Number median analytic radius	0.2986	μm
`--rds_swa_mcr`	Surface area weighted mean radius analytic	`cmd_ln_dfl`	μm
`--rds_vma_mcr`	Volume median radius analytic	`cmd_ln_dfl`	μm
`--rgh_mmn_dps_cmd_ln`	Roughness length momentum	`cmd_ln_dfl`	m
`--rgh_mmn_ice_std`	Roughness length over sea ice	0.0005	m
`--rgh_mmn_mbl`	Roughness length momentum for erodible surfaces	100.0×10^{-6}	m

Table 5: (continued)

Switch	Purpose	Default	Units
--rgh_mmn_smt	Smooth roughness length	10.0×10^{-6}	m
--sfc_typ	LSM surface type (0..28)	2	Index
--slr_cst	Solar constant	1367.0	W m^{-2}
--slr_spc_key	Solar spectrum string	“LaN68”	String
--slr_zen_ngl_cos	Cosine solar zenith angle	1.0	Fraction
--snw_hgt_lqd	Equivalent liquid water snow depth	0.0	m
--soi_typ	LSM soil type (1..5)	1	Index
--spc_heat_aer	Specific heat capacity	0.0	J kg^{-1} K^{-1}
--ss_alb_cmd_ln	Single scattering albedo	1.0	Fraction
--sz_dbg_mcr	Debugging size	1.0	μm
--sz_grd_sng	Type of size grid	“logarithmic”	String
--sz_mnm_mcr	Minimum size in distribution	0.9	μm
--sz_mxm_mcr	Maximum size in distribution	1.1	μm
--sz_nbr	Number of particle size bins	1	Number
--sz_prm_rsn	Size parameter resolution	0.1	Fraction
--tm_dlt	Timestep	1200.0	s
--tpt_bbd_wgt	Blackbody temperature of radiation	273.15	K
--tpt_gnd	Ground temperature	300.0	K
--tpt_ice	Ice temperature	tpt_frz_pnt	K
--tpt_mdp	Environmental temperature	300.0	K
--tpt_prt	Particle temperature	273.15	K
--tpt_soi	Soil temperature	297.0	K
--tpt_sst	Sea surface temperature	300.0	K
--tpt_vgt	Vegetation temperature	300.0	K

Table 5: (continued)

Switch	Purpose	Default	Units
`--tst_nm`	Name of test to perform	“”	String
`--var_ffc_gmm`	Effective variance of Gamma distribution	1.0	Fraction
`--vlm_frc_mntl`	Fraction of volume in mantle	0.5	Fraction
`--vmr_CO2`	Volume mixing ratio of CO_2	355.0×10^{-6}	molecule molecule^{-1}
`--vmr_HNO3_gas`	Volume mixing ratio of gaseous HNO_3	0.05×10^{-9}	molecule molecule^{-1}
`--vwc_sfc`	Volumetric water content	0.03	m^3 m^{-3}
`--wbl_shp`	Weibull distribution shape parameter	2.4	Fraction
`--wnd_frc_dps_cmd_ln`	Friction speed	`cmd_ln_dfl`	m s^{-1}
`--wnd_mrd_mdp`	Surface layer meridional wind speed	0.0	m s^{-1}
`--wnd_znl_mdp`	Surface layer zonal wind speed	10.0	m s^{-1}
`--wvl_dbg_mcr`	Debugging wavelength	0.50	μm
`--wvl_grd_sng`	Type of wavelength grid	“regular”	String
`--wvl_dlt_mcr`	Bandwidth	0.1	μm
`--wvl_mdp_mcr`	Midpoint wavelength	`cmd_ln_dfl`	μm
`--wvl_mnm_mcr`	Minimum wavelength	0.45	μm
`--wvl_mxm_mcr`	Maximum wavelength	0.55	μm
`--wvl_nbr`	Number of output wavelength bands	1	Number
`--wvn_dlt_xcm`	Bandwidth	1.0	cm^{-1}
`--wvn_mdp_xcm`	Midpoint wavenumber	`cmd_ln_dfl`	cm^{-1}
`--wvn_mnm_xcm`	Minimum wavenumber	18182	cm^{-1}
`--wvn_mxm_xcm`	Maximum wavenumber	22222	cm^{-1}
`--wvn_nbr`	Number of output wavenumber bands	1	Number

5 Appendix

5.1 Properties of Gaussians

The area under a Gaussian distribution may be expressed analytically when the domain is $(-\infty, +\infty)$. This result may be obtained (IIRC) by transforming to polar coordinates in the complex plane $x = r(\cos\theta + i\sin\theta)$.

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2/2}\, dx = \sqrt{2\pi} \tag{53}$$

This is a special case of a more general result

$$\int_{-\infty}^{+\infty} \exp(-\alpha x^2 - \beta x)\, dx = \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta^2}{4\alpha}\right) \qquad \text{where } \alpha > 0 \tag{54}$$

This result may be obtained by completing the square under the integrand, making the change of variable $y = x + \beta/2\alpha$, and applying (53). Substituting $\alpha = 1/2$ and $\beta = 0$ into (54) yields (53).

5.2 Error Function

The *error function* $\mathrm{erf}(x)$ may be defined as the partial integral of a Gaussian curve

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{-x^2}\, \mathrm{d}x \tag{55}$$

Using (53) and the symmetry of a Gaussian curve, it is simple to show that the error function is bounded by the limits $\mathrm{erf}(0) = 0$ and $\mathrm{erf}(\infty) = 1$. Thus $\mathrm{erf}(z)$ is the cumulative probability function for a normally distributed variable z (???). Most compilers implement $\mathrm{erf}(x)$ as an intrinsic function. Thus $\mathrm{erf}(x)$ is used to compute areas bounded by finite lognormal distributions (§3.1.3).