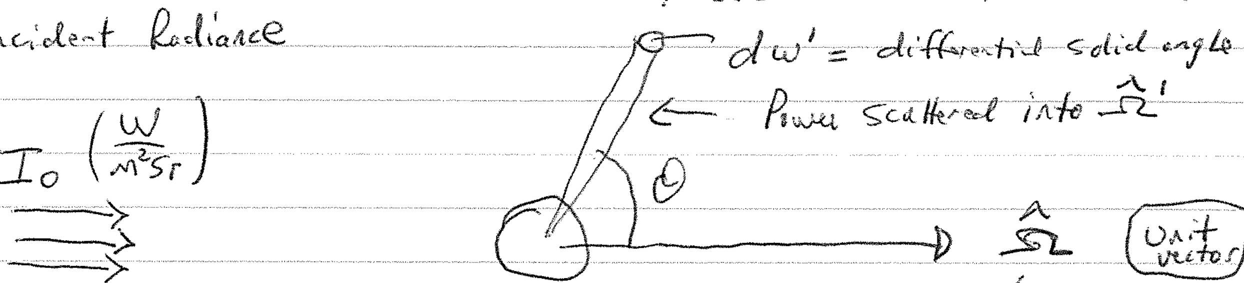
Ch 11. 1) General Equation For Radiative Transfer.
2) Scattering phase function.
3) Single scattering approximation.

Prelude to the Bohren paper model.

Incident Radiance

$I_0 \left(\frac{W}{m^2 sr}\right)$



Particle total Scattering Cross Section.

$$P_{sca}(\hat{\Omega}') = I_0 \frac{\sigma_{sca}}{4\pi} p(\hat{\Omega}, \hat{\Omega}') d\omega'$$

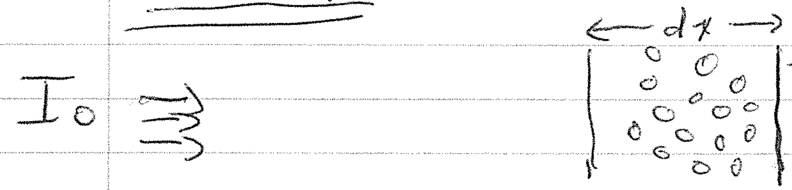
Solid angle

Scattering Phase Function

Solid angle of sphere

Normalization: $\frac{1}{4\pi} \int_{4\pi} p(\hat{\Omega}', \hat{\Omega}) d\omega' = 1$

Monodispersion: N identical scatterers / volume



particles = $N\,dx\,dA$

$$P_{sca}(\hat{\Omega}') = I_0 \underbrace{N\,dA\,dx\,\sigma_{sca}}_{} \frac{1}{4\pi} p(\hat{\Omega}, \hat{\Omega}') d\omega' \quad \beta_{sca}$$

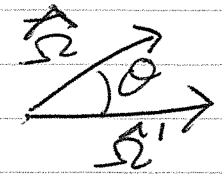
So $d\tau_{sca} = N \sigma_{sca}\, dx$

$$I_{sca}(\hat{\Omega}') = \frac{P_{sca}(\hat{\Omega}')}{dA} = I_0 \frac{d\tau_{sca}}{4\pi} p(\hat{\Omega}, \hat{\Omega}') d\omega'$$

$\Rightarrow$ Force $d\tau_{sca} <<< 1$ for single scattering.

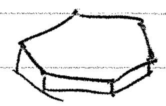
Example of Phase Functions:

For spherical particles or randomly oriented non spherical particles (like aerosols):



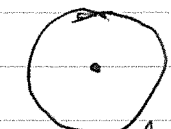
$P(\theta)$ only $\Rightarrow$ $\hat{\Omega}\cdot\hat{\Omega}' = \cos\theta$

[counter examples are cirrus ice crystals that are large enough to have a prefered orientation as they fall.]



← Plate ice crystal.

$g=0$ Isotropic Phase function: $P = \frac{1}{4\pi}$



Radiation Pattern

Rayleigh Phase function for natural light:

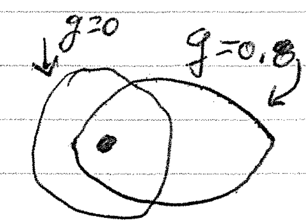
$g=0$ $$P(\theta) = \frac{3}{4}\left(1+\cos^2\theta\right)$$ Eq. 12.10 of GP.

Asymmetry Parameter $$g \equiv \frac{1}{4\pi}\int_0^{\pi} P(\theta)\cos\theta\, \underbrace{2\pi\sin\theta\, d\theta}_{d\omega}$$

π [Back Scattering]

0 [Forward Scattering]

Henyey Greenstein Phase Function:



$g=g$ $$P_{HG}(\theta) = \frac{1-g^2}{\left(1+g^2-2g\cos\theta\right)^{3/2}}$$

Forward Scatter: $P(\theta) = \frac{1}{4\pi}\delta(\theta)$

P such that $\delta(\theta) = \infty,\ \theta=0$

$= 0,\ \theta \neq 0$

Expansion of Phase function in Legendre Polynomials :

Symmetric — no ϕ dependence — randomly oriented particles

$$p(\cos\theta) = \sum_{l=0}^{\infty} \beta_l P_l(\cos\theta)$$

P_l — l th Legendre Polynomial

β_l — Expansion coefficients.

CONS — Complete, Orthogonal, Normal, Set

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = 1/2\,(3x^2 - 1)$$
$$P_3(x) = 1/2\,(5x^3 - 3x)$$
$$P_4(x) = 1/8\,(35x^4 - 30x^2 + 3)$$
$$P_5(x) = 1/8\,(63x^5 - 70x^3 + 15x)$$

orthogonality Condition

$$\int_{-1}^{1} P_n(x)\,P_m(x)\,dx = \begin{cases} 0, & n \neq m \\ \frac{2}{2n+1}, & n = m \end{cases}$$

$$\beta_l = \frac{2n+1}{2}\int_{-1}^{1} P_l(\underset{\mu}{\cos\theta})\; p(\underset{\mu}{\cos\theta})\; \underset{d\mu}{d\cos\theta}$$

So
$$\beta_0 = \frac{1}{2}\int_{-1}^{1} 1\;\underbrace{p(\mu)\,d\mu}_{=2} = 1$$

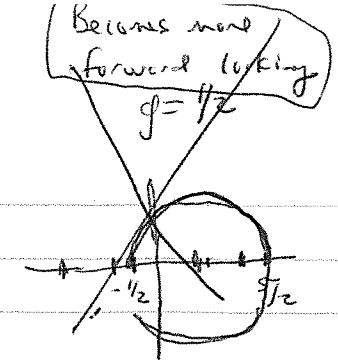
from Normalization of Phase function

$$\beta_1 = \frac{3}{2}\int_{-1}^{1} \cos\theta\; p(\cos\theta)\; d(\cos\theta)$$

But
$$g \equiv \frac{1}{2}\int_{-1}^{1} \cos\theta\; p(\cos\theta)\; d(\cos\theta)$$

So $\beta_1 = 3g$

Pg 1.7



The first two terms give

(1) $$p(\cos\theta) \cong 1 + 3g\cos\theta$$

Homework Problem: Determine the first 3 expansion coefficients, $Beta_\ell = \beta_\ell$, $\ell = 0, 1, 2$, for the Rayleigh phase function

$$p_{Rayleigh}(\cos\theta) = \frac{3}{4}(1 + \cos^2\theta)$$

Hint: You can work out a combination of the first three Legendre polynomials that give the Rayleigh phase function.

Aside Check normalization of (1): $1 = \frac{1}{2}\int_{-1}^{1}(\underbrace{1}_{\text{even term}} + \overbrace{3g\mu}^{\text{odd term}})d\mu$ ✓ checks out

(1) makes no sense if $p(\cos\theta) < 0$. Set $\theta = \pi$ $p(\cos\pi) = 1 - 3g \Rightarrow g < 1/3$

$-1/3 < g < 1/3$

It seems eq (1) and normalization require that

$$\sum_{\substack{\ell=2\\=1}}^{\infty} \int_{-1}^{1} \beta_\ell P_\ell(\mu)\, d\mu = 0$$

The odd terms are trivially satisfied. The even terms will go something like,

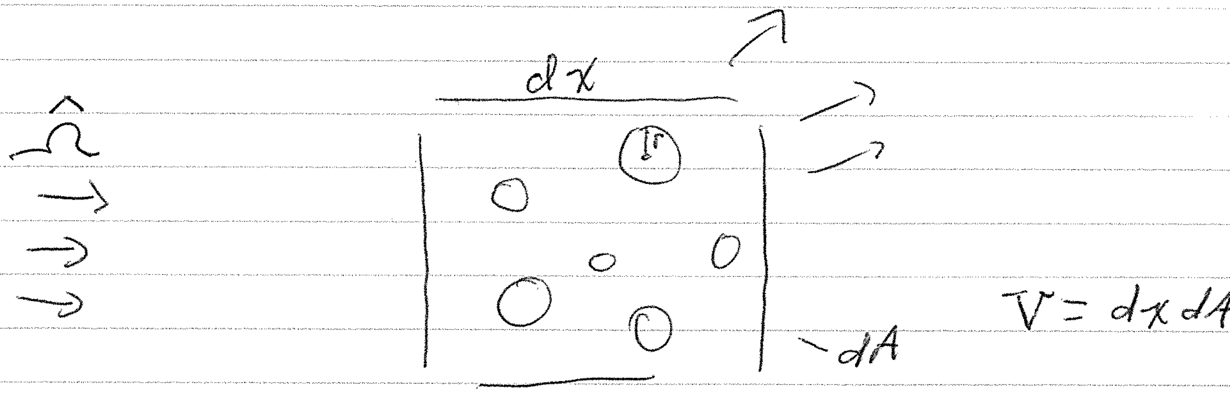
$$\int_{-1}^{1} d\mu\, P_2(\mu) = \int_{-1}^{1} 1/2\,(3\mu^2 - 1)\, d\mu = 1/2\left[\mu^3 - \mu\right]_{-1}^{1} = 0$$

And this is just a statement of orthogonality since $\beta_\ell = P_0 \beta_\ell$

Homework Problem

Use the two term phase function $1 + 3g\mu$ to compute the downwelling irradiance in the single scattering approximation as we did for the Rayleigh phase function. Interpret your results. [use the small optical depth approximation].

Scattering by a Polydispersion of Scatterers



$$I_{sca}(\hat{\Omega}') = \left[\frac{I_0\, dx}{4\pi}\right] \sum_{i=1}^{\#\text{Particles}} \sigma_{sca}(r_i,\lambda)\, P(r_i,\lambda;\hat{\Omega},\hat{\Omega}')$$

$$= [\;] \int_0^\infty \frac{dN}{dr}\, \sigma_{sca}(r,\lambda)\, P(r,\lambda;\hat{\Omega},\hat{\Omega}')\, dr$$

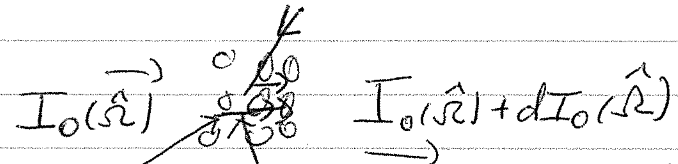
Define: Phase function for Polydispersion:

$$\bar{P}(\lambda;\hat{\Omega},\hat{\Omega}') = \frac{1}{4\pi} \frac{\int_0^\infty \frac{dN}{dr}\, \sigma_{sca}(r,\lambda)\, P(r,\lambda;\hat{\Omega},\hat{\Omega}')\, dr}{\underbrace{\int_0^\infty \frac{dN}{dr}\, \sigma_{sca}(r,\lambda)\, dr}_{=\beta_{sca}}}$$

$$I_{sca} = I_0 \frac{d\tau_{sca}}{4\pi} \bar{P}(\lambda;\hat{\Omega},\hat{\Omega}')$$

Then turn arrows around:

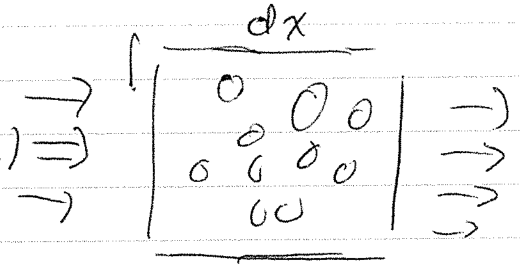
$$d I_0(\hat{\Omega}) = \frac{d\tau_{sca}}{4\pi} \int_{4\pi} I_0(\hat{\Omega}')\, \bar{P}(\lambda;\hat{\Omega},\hat{\Omega}')\, d\Omega'$$



Radiation from all directions that gets scattered into $\hat{\Omega}$.

Extinction and Emission Processes.

extinction



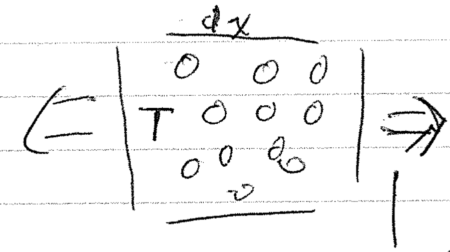
$I_0(\hat{\Omega}) \Rightarrow \quad \Rightarrow I_0 - \frac{P_{ext}}{dA}$

(Power removed by extinction in all directions.

$$\frac{P_{ext}}{dA} = \underbrace{N \sigma_{ext}\, dx}_{\tau_{ext}}\, I_0$$

Absorption

T = temperature



$$d\tau_{abs} = N \sigma_{abs}\, dx \ll 1$$

emissivity = absorption o.d.

$$d\tau_{abs}\, B(T,\lambda)$$

Radiance emitted by a black body of temperature T

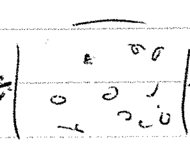
All Processes:

$$dI(\hat{\Omega}) = -d\tau_{ext}\, I(\hat{\Omega}) + d\tau_{abs}\, B(T,\lambda) + \frac{1}{4\pi}\, d\tau_{sca} \int_{4\pi} I(\hat{\Omega}')\, P(\hat{\Omega},\hat{\Omega}')\, d\Omega'$$

Extinction — removes power from forward beam

Thermal emission — adds power

(sum of scattering from all directions $\hat{\Omega}'$ that gets scattered into direction $\hat{\Omega}$.

$I(\hat{\Omega}) \rightarrow$  $\rightarrow I(\hat{\Omega}) + dI(\hat{\Omega})$

Use: $d\tau \equiv d\tau_{ext}$ $\quad d\tau_{sca} = \bar{\omega}\, d\tau_{ext}$

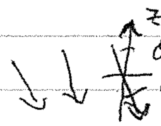
SSA

$d\tau_{abs} = (1-\bar{\omega})\, d\tau_{ext}$

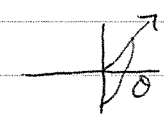
$$\Longrightarrow \boxed{\frac{dI(\hat{\Omega})}{d\tau} = -I(\hat{\Omega}) + (1-\bar{\omega})B(T,\lambda) + \frac{\bar{\omega}}{4\pi}\int_{4\pi} I(\hat{\Omega}')P(\hat{\Omega},\hat{\Omega}')\,d\omega'}$$

Formal equation for radiation transfer: Note it is an integro-differential equation ⇒ with a thermal emission source term!!

__Example Plane Parallel Approximation:__



Sunlight sets angle θ_0,



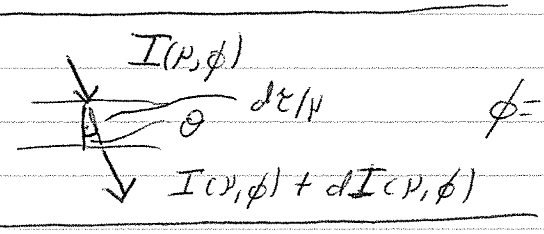
example of upward radiance

$\mu \equiv \cos\theta$

$d\mu = -\sin\theta\, d\theta$

$\tau = 0$ TOA

$d\tau$



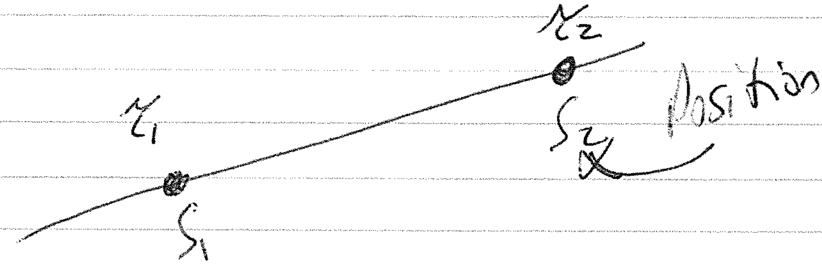
ϕ = azimuthal angle

$\tau = \tau_1$

$$\mu\frac{dI(\mu,\phi)}{d\tau} = -I(\mu,\phi) + \frac{\bar{\omega}}{4\pi}\int_0^{2\pi}\int_{-1}^{1} P(\mu,\phi;\mu',\phi')\,I(\mu',\phi')\,d\mu'\,d\phi'$$

Since $d\omega' = \sin\theta\, d\theta\, d\phi = -d\mu\, d\phi$

Formal Solution:



Notice that

$$\frac{d}{d\tau}\left\{ I(\hat{\Omega})\, e^{\tau} \right\} = e^{\tau}\frac{dI}{d\tau} + I e^{\tau}$$

$$= e^{\tau}\left[\frac{dI}{d\tau} + I \right]$$

So we can rewrite the general solution from page 5 as

$$\frac{d}{d\tau}\left\{ I(\hat{\Omega})\, e^{\tau} \right\} = (1-\bar{\omega})B(T,\lambda)e^{-\tau} + e^{-\tau}\frac{\tilde{\omega}}{4\pi}\int_{4\pi} I(\hat{\Omega}')\bar{P}(\hat{\Omega},\hat{\Omega}')\,d\omega'$$

Let $J(T,\hat{\Omega},\hat{\Omega}',\lambda,\bar{\omega}) \equiv$ Radiation Source term [from emission and scattering].

$$J \equiv (1-\bar{\omega})B(T,\lambda) + \frac{\bar{\omega}}{4\pi}\int_{4\pi} I(\hat{\Omega}')\bar{P}(\hat{\Omega},\hat{\Omega}')\,d\omega'$$

$$d\left\{ I e^{\tau} \right\} = e^{-\tau} J$$

Integrating:

$$\int_{I_1 e^{\tau_1}}^{I_2 e^{\tau_2}} d\{I e^{\tau}\} = \int_{\tau_1}^{\tau_2} e^{-\tau} J \, d\tau$$

So

$$\boxed{I_2 = I_1 e^{-(\tau_2-\tau_1)} + e^{-(\tau_2-\tau_1)} \int_{\tau_1}^{\tau_2} e^{-\tau} J \, d\tau}$$

direct Beam

Radiation Sources

This is more of a symbolic solution because J has I in it as well.

Note: in general, along the path from S_1 to S_2,

$$d\tau(s) = \beta_{ext}(s)\, ds$$

<u>Example</u> $\longrightarrow \bigcirc \longrightarrow \quad \bar{P} = \delta(\mu-\mu_0)\,\delta(\phi-\phi_0)\,4\pi$

[perfect forward scattering] $\Rightarrow$ <u>No scattering</u>

$$\mu \frac{dI(\mu,\phi)}{d\tau} = -I(\mu,\phi) + \bar{\omega}\, I(\mu,\phi)$$

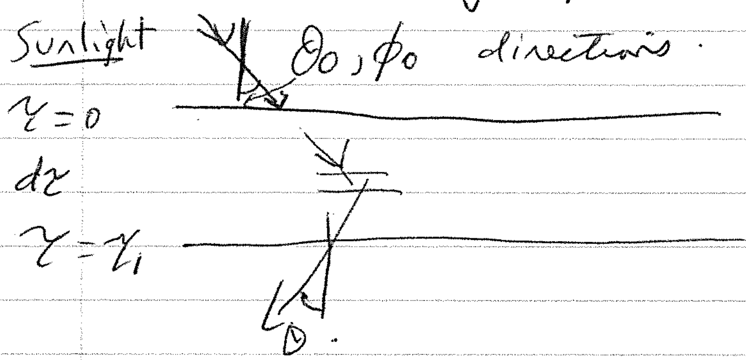
$$= -I(\mu,\phi)\,(1-\bar{\omega})$$

$$\left(d\tau_{abs} = (1-\bar{\omega})\,d\tau \right.$$

or

$$\mu \frac{dI(\mu,\phi)}{d\tau_{abs}} = -I(\mu,\phi)$$

<u>Single Scattering approximation:</u>



Let $I(\mu',\phi') \equiv F_0\,\delta(\mu'-\mu_0)\,\delta(\phi'-\phi_0)\,e^{-\tau/\mu_0}$

Then from pg 5,

(1) $$\mu \frac{dI(\mu,\phi)}{d\tau} = -I(\mu,\phi) + \frac{F_0\tilde{\omega}}{4\pi}\,\bar{P}(\cos\Theta)\,e^{-\tau/\mu_0}$$

a. Multiply by $e^{\tau/\mu}$ on both sides.

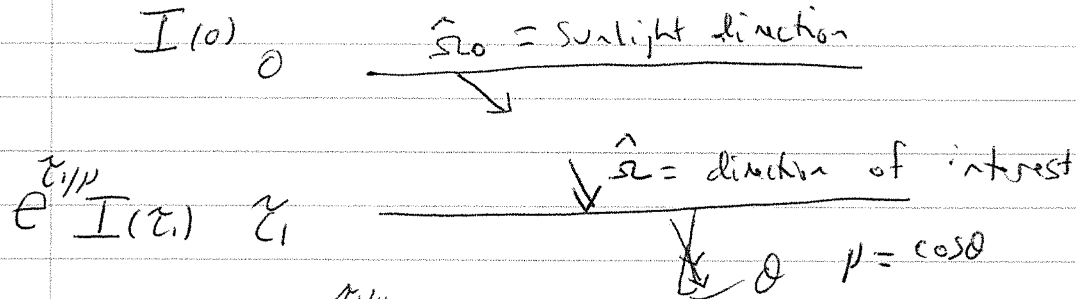
b. Divide by μ.

(2) $$\frac{dI}{d\tau}e^{\tau/\mu} + \frac{I}{\mu}e^{\tau/\mu} = \frac{F_0\tilde{\omega}}{4\pi\mu}\,\bar{P}\,e^{-\tau/\mu_0}e^{+\tau/\mu}$$

Note: $$\frac{d}{d\tau}\left[I e^{\tau/\mu}\right] = \frac{dI}{d\tau} e^{\tau/\mu} + \frac{1}{\mu} I e^{\tau/\mu}$$

So

$$\frac{d}{d\tau}\left[I e^{\tau/\mu}\right] = \frac{F_0 \tilde{\omega}}{4\pi\mu} \bar{P} \; e^{-\tau(1/\mu_0 - 1/\mu)}$$



$$\int_{I(0)}^{e^{\tau_1/\mu} I(\tau_1)} d\left[I e^{\tau/\mu}\right] = \frac{F_0 \tilde{\omega} \bar{P}}{4\pi\mu} \int_0^{\tau_1} e^{-\tau(1/\mu_0 - 1/\mu)} d\tau$$

or

$$\boxed{e^{\tau_1/\mu} I(\tau_1) - I(0) = \frac{F_0 \tilde{\omega} \bar{P}}{4\pi\mu} \frac{1}{1/\mu - 1/\mu_0}\left[e^{-\tau_1(1/\mu_0 - 1/\mu)} - 1\right]}$$

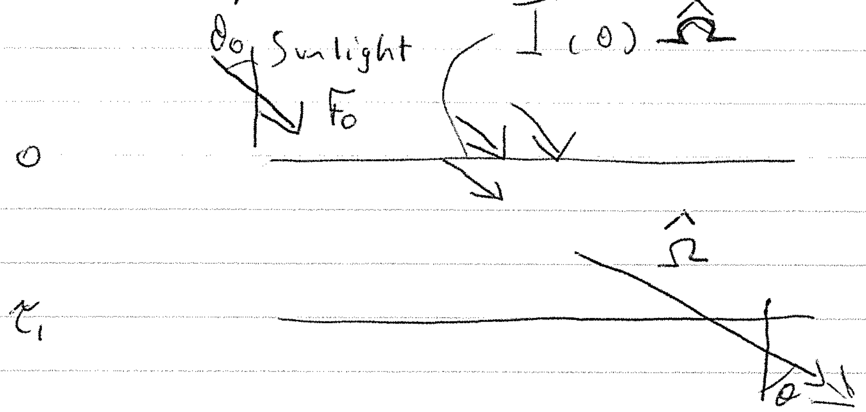
Then

$$\boxed{I(\tau_1) = I(0) e^{-\tau_1/\mu} + \frac{F_0 \tilde{\omega} \bar{P}}{4\pi\mu} \frac{\left[e^{-\tau_1/\mu_0} - e^{-\tau_1/\mu}\right]}{1/\mu - 1/\mu_0}}$$

If $\tau_1/\mu_0 + \tau_1/\mu \ll 1$ $\quad e^{-\tau_1/\mu_0} - e^{-\tau_1/\mu} \approx 1 - \tau_1/\mu_0 - 1 + \tau_1/\mu$

$$I(\tau_1, \theta) \cong I(0) e^{-\tau_1/\cos\theta} + \frac{F_0 \tilde{\omega} \bar{P}_1(\cos(\theta))}{4\pi \cos\theta} \tau_1$$

Interpretation:



$\Theta = \theta - \theta_0$

We could take $I(0) = F_0\, \delta(\mu - \mu_0)\, \delta(\phi - \phi_0)$

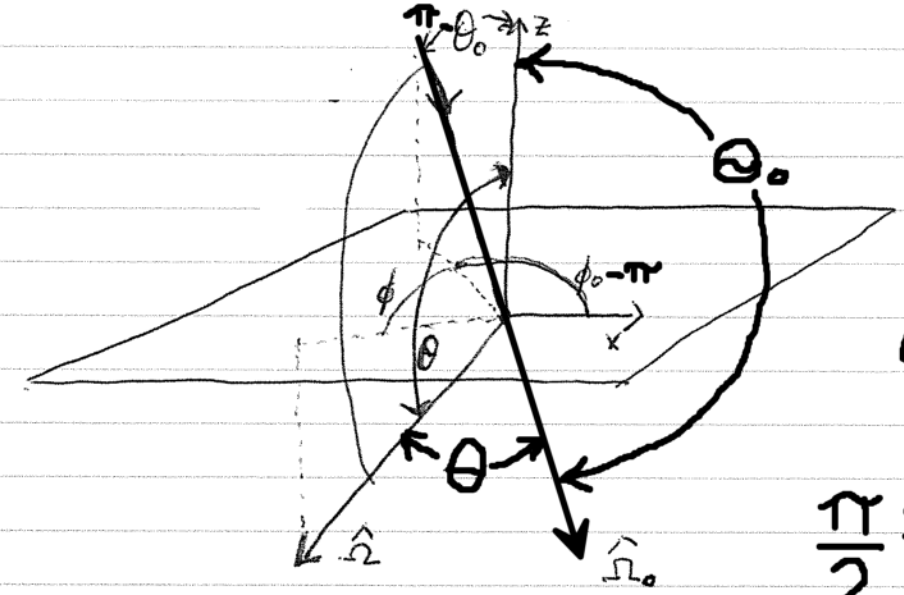
Solar Zenith   Solar Azimuth

Then the direct beam solution is, for a finite acceptance angle

I

$$x = r\sin\theta\cos\phi \qquad \hat{x} = \sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi \qquad \hat{y} = \sin\theta\sin\phi$$
$$z = r\cos\theta \qquad \hat{z} = \cos\theta$$

draw ϕ differently



$$0 \le \phi \le 2\pi$$

$$\frac{\pi}{2} \le \Theta_0 \le \pi$$

Then

$$\hat{\Omega}_0 = (\sin\theta_0\cos\phi_0,\ \sin\theta_0\sin\phi_0,\ \cos\theta_0)$$
$$\hat{\Omega} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

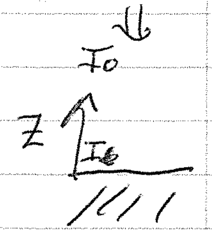
Use $\cos(a-b) = \cos a\cos b - \sin a \sin b$

$$\hat{\Omega}_0 \cdot \hat{\Omega} = \cos\Theta = \sin\theta_0\cos\phi_0\sin\theta\cos\phi + \sin\theta_0\sin\phi_0\sin\theta\sin\phi + \cos\theta\cos\theta_0$$

$$= \sin\theta_0\sin\theta\,[\cos\phi_0\cos\phi + \sin\phi_0\sin\phi] + \cos\theta\cos\theta_0$$

$$\boxed{\cos\Theta = \sin\theta_0\sin\theta\cos(\phi_0-\phi) + \cos\theta_0\cos\theta}$$

Calculate the diffuse downwelling Flux from Rayleigh Sca.



$I = I_o e^{-\tau_1}$

τ_1 = total normal optical depth due to Rayleigh Sca.

$$I(\tau_1, \theta, \phi) \cong \frac{F_o \tilde{\omega} \bar{P}[\cos\Theta]\, \tau_1}{4\pi \cos\theta} \quad \forall \tau_1 \ll 1$$

Rayleigh Phase Function: $\bar{P}[\cos\Theta] = \frac{3}{4}(1+\cos^2\Theta)$

Downwelling Irradiance — Note these θ limits!! Need ↓

$$F_\downarrow = \int_0^{2\pi}\int_{\pi/2}^{\pi} I(\tau_1,\theta,\phi)\cos\theta\sin\theta\, d\theta\, d\phi$$

Let $\mu = \cos\theta$ $\quad d\mu = -\sin\theta\, d\theta$. Note $\tilde{\omega} = 1$.

$\sin\theta = \pm\sqrt{1-\mu^2}$

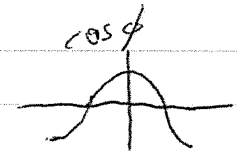
$$F_\downarrow = \left[\frac{3}{4}\frac{F_o \tau_1}{4\pi}\right]\int_0^{2\pi}\int_{-1}^{0}\left(1 + \left\{\mu\cos\theta_o + \sqrt{1-\mu^2}\sin\theta_o\cos(\phi-\phi_o)\right\}^2\right] d\mu\, d\phi$$

First term $= 2\pi$

2nd term

$$F_\downarrow = [\;]\cdot 2\pi + [\;]\int_0^{2\pi}\int_{-1}^{0}\Big[\mu^2\cos^2\theta_o + 2\mu\sqrt{1-\mu^2}\sin\theta_o\cos\theta_o\cos(\phi-\phi_o) + (1-\mu^2)\sin^2\theta_o\cos^2(\phi-\phi_o)\Big]\, d\mu\, d\phi$$

Thankfully



$$\int_0^{2\pi}\cos\phi\, d\phi = 0$$

And $\quad \cos^2(\phi-\phi_o) = \frac{1}{2}\cos[2(\phi-\phi_o)] + \frac{1}{2}$

So

$$F_\downarrow = 2\pi[\;] + 2\pi[\;]\int_{-1}^{0}\left(\mu^2\cos^2\theta_o + (1-\mu^2)\frac{\sin^2\theta_o}{2}\right)d\mu$$

So

$$F_\downarrow = 2\pi[\ ] + 2\pi[\ ]\left(\frac{\cos^2\theta_0}{3} + \frac{\sin^2\theta_0}{2}\underbrace{\int_{-1}^{0}(1-\mu^2)\,d\mu}_{1-1/3=2/3}\right)$$

Using $\cos^2\theta_0 + \sin^2\theta_0 = 1$,

$$F_\downarrow = 2\pi\left[\ \right]\left(1+\frac{1}{3}\right)$$

$$= 2\pi \cdot \frac{3}{4}\,\frac{F_0\tau_1}{4\pi}\cdot\frac{4}{3}$$

$$\boxed{F_\downarrow = \frac{F_0\tau_1}{2}}$$

Downwelling diffuse radiation due to Rayleigh scattering.

Still a distribution function

$F_0(\lambda)$

$\tau_1(\lambda)$

F_0 $\frac{W}{m^2\cdot nm}$

Works well for τ_1 and $\lambda > 500$ nm.

Note

$F\uparrow$ TOA

$F\downarrow$

//////

$F\uparrow = F\downarrow$ if Surface Albedo $= \underline{0}$.