where the subscript N denotes components normal to the path of the particle, and the subscripts R and θ denote components in the radial direction and perpendicular to the radius, respectively.

As to the origin or nature of this transverse force produced by motion, it is of interest to observe that it is similar to the "fundamental law" proposed in a posthumous note by Gauss[6] for the mutual action of two elements of electricity in relative motion. The occurrence of c^2 stems from the idea of the attraction being transmitted with the speed of light.

The Birkhoff force equations for a planetary orbit can be summarized, according to this analysis, as follows: They are the equations one would obtain from a Newtonian attractive force acting on the Lorentzian local mass of the planet, with the addition of forces caused by motion, transverse to the path of the planet, which do not affect the conservation of energy but alter the areal constant. The precise value of these added forces and the method of obtaining them is thus of crucial importance. Apparently these forces are not introduced by Birkhoff with conscious resort to physical concepts, but they are present because of his choice for gravity of the third term[2] of a formal expansion in rational and integral components of a typical force function, in which successive terms are of increasing complexity and hence provide for additional force components. An independent physical derivation of these transverse forces would be welcome.

[6] C. F. Gauss, *Werke* (Göttingen, 1863–74), Vol. 5, p. 616.

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Refraction of Plane Non-Uniform Electromagnetic Waves between Absorbing Media

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When a plane non-uniform electromagnetic wave is refracted between two conducting media, there are two possible positions for the propagation vector in the second medium. Consideration of the energy flow shows that each solution holds within a certain range of values of the (complex) angle of incidence, the transition from one to the other occurring in a discontinuous way. The two cases of the electric vector perpendicular and parallel to the plane of incidence are discussed.

(1) INTRODUCTION

The problem of the refraction of a plane non-uniform electromagnetic wave at the plane boundary between two conducting media is not generally fully discussed in textbooks where it is pointed out that, with the use of complex angles of incidence and refraction and of complex propagation vectors, the problem is formally identical to the usual one in which perfect dielectrics are involved.[1]

It is the purpose of this paper to complete this treatment discussing the new physical features which appear when both media are conducting, in particular, a discontinuity occurring in the (complex) propagation vector in the second medium at the (complex) angle of incidence for which there is no average flow of energy across the boundary.

(2) THE PROPAGATION VECTOR IN THE SECOND MEDIUM

Let the boundary be the plane $y-z$, and the plane of incidence the plane $x-z$, the x-axis being directed from the first medium into the second. Let any field component be represented by

$$Ee^{-\mathbf{k}_i\cdot\mathbf{r}+i\omega t} \tag{1}$$

with

$$\mathbf{k}_1=\mathbf{a}+i\mathbf{b};\quad \mathbf{k}_2=\mathbf{A}+i\mathbf{B}, \tag{2}$$

E being a complex amplitude.

[1] See, for instance, J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), pp. 500–524.

Given $\mathbf{k}_1$ and the constants of the second medium, $\mathbf{k}_2$ is determined by:

(a) The relations holding generally for non-uniform waves[2]

$$k_2{}^2 = A^2 - B^2 + 2i\mathbf{A}\cdot\mathbf{B} = \omega^2\mu\epsilon_2 + i\omega\mu\sigma_2, \qquad (3)$$

where all letters have their usual meaning, and μ will be assumed to be the same for both media;

(b) The boundary conditions, which give (Snell's law):

$$k_1 \sin\theta_1 = k_2 \sin\theta_2, \qquad (4)$$

θ_1, θ_2 being the (complex) angles of incidence and refraction. Equation (4) is equivalent to the two relations between real quantities:

$$a_z = A_z; \quad b_z = B_z; \qquad (5)$$

a_z and b_z can be expressed as functions of the constants of the first medium and of θ_1; it is convenient in the following to consider them as the independent variables in terms of which to express the propagation vector in the second medium.

Conditions (3) and (5) are not sufficient to determine this vector in a unique way. In fact, if

$$2A_zB_z = 2a_zb_z > \omega\mu\sigma_2, \qquad (6)$$

the product A_xB_x is negative, but, without consideration of the energy flow, we cannot immediately decide which of the two factors is negative. Only when A_xB_x is positive, physical reasons will immediately rule out the solution with both quantities negative. Even in the case of perfect dielectrics it is possible formally to construct a second solution for the field in the second medium, satisfying Maxwell's equations and the boundary conditions and corresponding to a "refracted wave" moving towards the boundary and on the same side of the normal as the incident wave: this solution is absurd from a physical point of view. It is an example of the striking way in which the presence of a conductivity in both media modifies these phenomena, that in our case a decision between the two possible solutions is not so immediate, and that, as we shall see, the requirement that the energy should flow from medium 1 to medium 2 leads to the necessity of choosing one solution within a certain range of the complex angle of incidence and the other within another range, the transition occurring in a discontinuous way.

[2] See, for instance, J. A. Stratton, reference 1, p. 341.

(3) THE ENERGY FLOW

(a) Electric vector Perpendicular to the Plane of Incidence ($E_\perp$)

If E, E', E'' are the complex amplitudes of the electric vector in the incident, reflected, and transmitted waves, respectively, the average flow of energy perpendicular to the boundary in the first medium is given by:

$$\langle S_x{}^{(1)}\rangle_{\text{Av}} = Re(\tfrac{1}{2}E_yH_z{}^*)$$
$$= (e^{-2a_2z}/2\mu\omega)[\,|E|^2b_xe^{-2a_xx} - |E'|^2b_xe^{2a_xx}$$
$$-2a_x|E|\,|E'|\,\sin(2b_xx+\gamma)], \qquad (x<0) \quad (7)$$

with

$$\gamma = \arg\,(E'/E). \qquad (8)$$

The first two terms may be interpreted as the energy flow associated with the incident and reflected waves, respectively. The third term,[3] which vanishes for $\sigma_1 = 0$, is caused by interference between the incident and reflected waves: there is a local radiation from the maxima to the minima of the standing waves system.

The average flow of energy perpendicular to the boundary in the second medium is:

$$\langle S_x{}^{(2)}\rangle_{\text{Av}} = (1/2\mu\omega)e^{-2(A_xx+A_zz)}|E''|^2B_x, \qquad (x>0) \quad (9)$$

so that the condition that S_x be continuous at $x=0$ is expressed by:

$$b_x[1-|E'/E|^2] - 2a_xIm(E'/E) = B_x|E''/E|^2. \qquad (10)$$

This condition is automatically satisfied by the expressions for E' and E'', whatever the sign of B_x; however the physical conditions of our problem impose that the energy flow be directed from medium 1 to medium 2, therefore the additional relation for the unique determination of $\mathbf{k}_2$ is in this case

$$B_x \geqslant 0. \qquad (11)$$

[3] M. Born and E. Ladenburg, Physik Zeits. 12, 198 (1911).

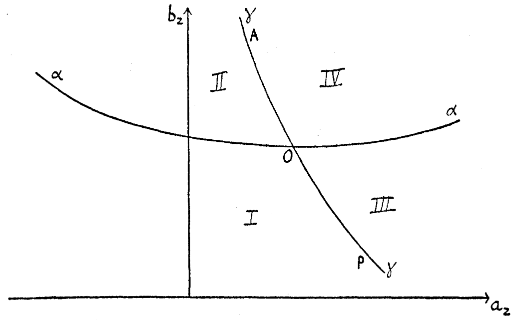

FIG. 1. E perpendicular to the plane of incidence. Discontinuity along AO.

(b) Electric Vector in the Plane of Incidence ($E_{\parallel}$)

In this case the condition of continuity of S_x at the boundary gives, if we indicate by H, H', H'', the amplitudes of the magnetic vector in the incident, reflected, and transmitted waves, respectively:

$$\frac{a_x\sigma_1+b_x\epsilon_1\omega}{\sigma_1^2+\epsilon_1^2\omega^2}(|H|^2+|H'|^2)+2\frac{b_x\sigma_1-a_x\epsilon_1\omega}{\sigma_1^2+\epsilon_1^2\omega^2}|H|\,|H'|\sin\gamma=\frac{A_x\sigma_2+B_x\epsilon_2\omega}{\sigma_2^2+\epsilon_2^2\omega^2}|H''|^2$$

$$\text{with} \qquad \gamma=\arg(H'/H), \qquad (12)$$

and the condition that the energy must flow in the right direction is expressed by

$$A_x\sigma_2+B_x\omega\epsilon_2\geqslant 0. \qquad (13)$$

(4) THE DISCONTINUITY FOR THE CASE $E_{\perp}$

Writing explicitly the equations determining A_x and B_x (see (3) and (5)):

$$B_x^2-A_x^2=\epsilon_2\mu\omega^2-b_z^2+a_z^2, \qquad (14)$$

$$2A_xB_x=\sigma_2\mu\omega-2b_za_z, \qquad (15)$$

we see that either A_x or B_x is zero when

$$2a_zb_z=\sigma_2\mu\omega, \qquad (16)$$

and that, when (16) is satisfied:

$$B_x \text{ is zero if } b_z^2\geqslant\epsilon_2\mu\omega^2+a_z^2, \qquad (17)$$
$$A_x \text{ is zero if } b_z^2\leqslant\epsilon_2\mu\omega^2+a_z^2.$$

In the case of small conductivities ($\sigma\ll\epsilon\omega$)

$$b_z\rightarrow(\epsilon_1\mu)^{\frac{1}{2}}\omega\sin\varphi_1, \qquad (18)$$

φ_1 being the real angle of incidence, and $a_z\rightarrow 0$, so that Eq. (17) becomes

$$\sin\varphi_1\gtreqless n, \quad \text{with} \quad n=(\epsilon_2/\epsilon_1)^{\frac{1}{2}}. \qquad (19)$$

Although the notion of critical angle loses all meaning in the case of conducting media, we shall for brevity say that, when (16) is satisfied, B_x vanishes in the region of total reflection, and A_x in the region of ordinary refraction.

If now we plot the curves (Fig. 1)

$$b_z^2-a_z^2=\epsilon_2\mu\omega^2 \text{ (line } \alpha), \qquad (20)$$

$$2a_zb_z=\mu\omega\sigma_2 \text{ (line } \gamma), \qquad (21)$$

and divide the plane a_zv_z into four regions (no loss of generality is entailed in excluding negative values of b_z), B_x vanishes on the boundary between regions II and IV, and A_x on the boundary between I and III. The product A_xB_x is negative in regions III and IV. As B_x must always be positive (Eq. 11), and no change of sign of either factor occurs in crossing line α (Eq. 14), it follows that A_x is positive everywhere left and negative everywhere right of line γ. In the region of ordinary refraction the transition from positive to negative A_x occurs continuously, passing through $A_x=0$, but in the region of total reflection A_x does not vanish on line γ; therefore it must be there discontinuous. Indeed it jumps from the value $+(b_z^2-a_z^2-\epsilon_2\mu\omega^2]^{\frac{1}{2}}$ on the left to the value $-[b_z^2-a_z^2-\epsilon_2\mu\omega^2]^{\frac{1}{2}}$ on the right—namely, the planes of equal amplitude in the second medium, perpendicular to $\mathbf{A}$, have, on the two sides of the discontinuity, symmetrical positions with respect to the boundary.

To the two different orientations of $\mathbf{k}_2$ correspond different expressions for the field components, and also for these the transition from one set of values to the other occurs discontinuously when $B_x=0$. Experimental verification of these considerations could be better obtained by investigation of this last phenomenon.

A negative value of A_x means that the amplitude increases exponentially for $x\rightarrow\infty$; but in an actual case the cross section of the beam is not infinite, and then the amplitude decreases in penetrating into the second medium, as follows from $\mathbf{A}\cdot\mathbf{B}>0$.

A difficulty arises from the fact that at the discontinuity ($B_x=0$) there is no average flow of energy across the boundary, and it is not apparent how the field in the second (dissipative) medium can be maintained. This difficulty is removed when a beam of finite cross section is considered, for then edge effects allow for some energy crossing the boundary even when $B_x=0$.[4]

(5) THE DISCONTINUITY FOR THE CASE $E_{\parallel}$

In this case the energy condition is:

$$(\sigma_2/\omega\epsilon_2)A_x+B_x \gtrless 0. \tag{22}$$

The presence of A_x in this expression is caused by the fact that, when the electric vector is in the plane of incidence, the Poynting vector, for a non-uniform wave, is not parallel to **B** (is not perpendicular to the planes of equal phase). In this case we have thus the possibility of B_x being negative, provided $\langle S_x^{(2)}\rangle_{\mathrm{Av}} \gtrless 0$.

Taking the equality sign in (22), and eliminating A_x, B_x from (14), (15), (22), we obtain the pairs of values of a_z and b_z for which the flow of energy across the boundary vanishes, and for which we can consequently expect a transition from one solution for A_x, B_x to the other with reversed signs. We obtain:

$$b_z^2-a_z^2+\left[\frac{\sigma_2}{\omega\epsilon_2}-\frac{\omega\epsilon_2}{\sigma_2}\right]a_zb_z=\tfrac{1}{2}\epsilon_2\mu\omega^2\left[1+\frac{\sigma_2^2}{\epsilon_2^2\omega^2}\right]. \tag{23}$$

If $\sigma_2<\epsilon_2\omega$, Eq. (23), is represented by a line such as line β in Fig. 2, where lines α and γ are the same as in Fig. 1. As before, the product A_xB_x is negative only right of the line γ; thus, since the three hyperbolae meet at 0, the important part of curve β is only that in the region of total reflection; the remaining part corresponds to a spurious solution. A_x, as before, vanishes along OP, and is negative right of BOP. B_x vanishes along AO and now becomes negative in the region AOB. Along curve β, where the average flow of energy across the boundary vanishes, we must pass from one solution to the other; there is thus a discontinuous transition from (substituting (22) in (15)):

$$\begin{aligned} B_x&=[(\sigma_2/2\omega\epsilon_2)(2a_zb_z-\mu\omega\sigma_2)]^{\frac{1}{2}},\\ A_x&=-[(\omega\epsilon_2/2\sigma_2)(2a_zb_z-\mu\omega\sigma_2)]^{\frac{1}{2}} \end{aligned} \tag{24}$$

to the solution with opposite signs.

[4] J. Picht, Ann. d. Physik **3**, 433 (1929); F. Noether, Ann. d. Physik **11**, 141 (1931).

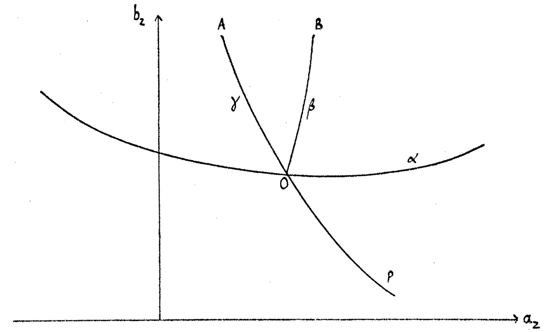

FIG. 2. E parallel to the plane of incidence. Discontinuity along BO.

Particular cases, and the relative position of the three hyperbolae for different values of the parameters, are easily discussed.

(6) USE OF THE REAL ANGLES OF INCIDENCE AS INDEPENDENT PARAMETERS

An alternative formulation of the above treatment may be given assuming as independent variables, instead of a_z and b_z, the real angles φ_1, ψ_1 that the normals to the planes of equal phase and of equal amplitude, respectively, make with the positive x-axis in the first medium. Formulae become then more expressive. Thus Eq. (21) representing the condition for the changing of sign of the product A_xB_x becomes

$$\cot\varphi_1 \cot\psi_1=(\sigma_1/\sigma_2)-1, \tag{25}$$

and the condition for the polarizing angle, which makes $H'=0$ in the case $E_{\parallel}$ may be expressed as

$$\cot\varphi_1 \cot\psi_2+\cot\varphi_2 \cot\psi_1=2, \tag{26}$$

φ_2, ψ_2 being the angles corresponding to φ_1 and ψ_1 in the second medium.

An experimental arrangement for the verification of the above theory would consist of a prism (large with respect to the wave-length) made of the material of medium 1, and bounded on the second face by medium 2. By varying the angle of incidence on the first face, and the angle of the prism, all values of φ_1 and ψ_1 can be obtained. Such an experiment may provide a new method for the determination of conductivities.