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Multiple scattering of light and some of its observable consequences

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Many common observations are inexplicable by single-scattering arguments: the variation of brightness and color of the clear sky; the brightness of clouds; the whiteness of a glass of milk; the appearance of distant objects; the blueness of light transmitted in snow and other natural ice bodies; the darkening of sand upon wetting. Yet multiple scattering is seldom mentioned in optics textbooks. It is possible to understand many observable phenomena without invoking the complete theory of multiple (incoherent) scattering. A simple two-stream theory, in which photons are constrained to be scattered in only two directions, forward and backward, is adequate for interpreting many observations, even quantitatively, and it paves the way for advanced study.

I. INTRODUCTION

Multiple scattering of light gives rise to observable phenomena that cannot be explained by single-scattering arguments. For example, if single scattering prevailed in the atmosphere the sky would be uniform in color, which is contrary to what is observed. Clouds are white and bright mostly because of multiple scattering. Attenuation of visible light by ice grains is not spectrally selective, and yet crevasses and ice caves and even holes in ordinary snow may display hues more vivid than those of the bluest sky. Milk is a suspension of small particles that scatter blue light more than red, and yet a glass of milk is white. A white sandy beach—or salt, or sugar—has properties not shared by its grains. And who has not noticed sand darken after being washed by waves, or soil darken when wet by rain?

Single-scattering arguments are insufficient to explain these common observations. Yet multiple scattering is seldom mentioned in optics textbooks. A student or teacher who would learn something about multiple scattering must consult monographs such as those by Chandrasekhar[1] or by van de Hulst.[2] Although these are invaluable for specialists, neophytes are likely to find them formidable: They emphasize techniques for solving equations rather than developing physical intuition by explaining observations using simple models.

Because of this lack of suitable introductory treatments of multiple light scattering, I offer the following. My purpose is to convey as much physical understanding as possible from the least amount of mathematics. Approximate equations, which can be solved exactly, are derived, rather than exact equations which can be solved only approximately. This provides a simple theoretical framework for interpreting many observations. As an intended side effect, the terms and concepts found in advanced treatises are introduced, thereby smoothing the way for further study.

II. TWO-STREAM EQUATIONS OF RADIATIVE TRANSFER

Any scattering medium is composed of discrete scatterers, be they molecular or particulate. Since it is inconvenient to consider this discreteness explicitly, we usually replace discrete media with hypothetical continuous media, the scattering and absorption properties of which are determined by those of the former. The resulting continuum theories are applicable to discrete media provided they contain a great many scatterers in any volume of interest.

Throughout this paper *incoherent* scattering is assumed, that is, we do not take into account phase differences between scattered waves. *Coherent* scattering is treated in Refs. 3–7. There is no sharp boundary between coherently and incoherently scattering media. There are merely different *approximate* theories, in some of which phases are taken into account and in others they are not, which are applied with varying degrees of success to the prediction and interpretation of observations. An ordinary cloud is a typical medium in which incoherent scattering dominates what is observed, whereas a glass of pure water is a medium in which coherent scattering dominates (see, e.g., Ref. 8 for a good discussion of the scattering interpretation of Fresnel's equations).

We also assume that radiation can be scattered in only *two* directions, forward and backward. Finally, polarization is ignored.[9] These simplifying assumptions keep the mathematics manageable without greatly distorting our picture of reality. They underlie the elementary two-stream theory first set down by Schuster.[10] Subsequently, other

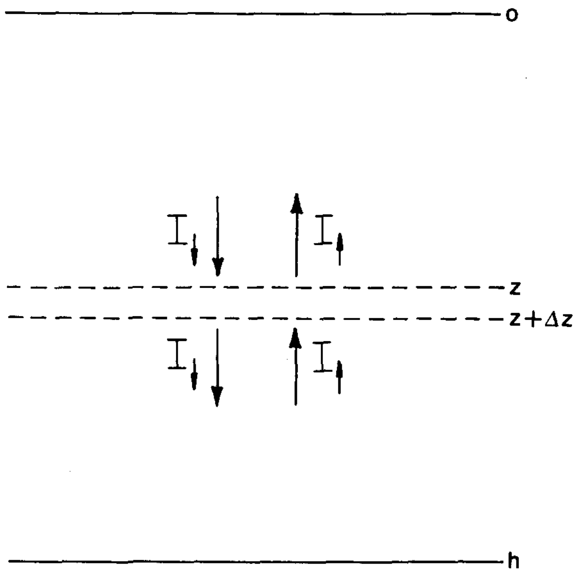

Fig. 1. Conservation of radiant energy applied to the region Δz in a scattering-absorbing medium of infinite lateral extent yields the two-stream equations of transfer for the downward and upward intensities $I_\downarrow$ and $I_\uparrow$.

two-stream equations have been derived (e.g., Refs. 11–13; see Refs. 14 and 15 for comparisons of various two-stream theories), usually by beginning with the integro-differential equation of radiative transfer and making various approximations. Our approach is to sidestep the exact equation and give a physical derivation similar to that of Schuster's.[10]

After a photon is emitted it suffers only one of two fates when it interacts with matter: (1) it is *absorbed*, that is, it ceases to exist, although its energy is taken up by whatever it interacts with; or (2) it is *scattered*, in which instance it survives the interaction intact but possibly changes direction. Consider a continuous scattering–absorbing medium, infinite in lateral extent, bounded by parallel planes (i.e., a *plane-parallel* medium). We assume that photons are emitted only by sources outside this medium. To derive the equations of radiative transfer for it we apply a radiant energy balance to a small region Δz (Fig. 1). Photons in a given direction incident on this region are lost by absorption and by scattering within it. But there is also a gain of photons because those in one direction are scattered into the opposite direction. By $I_\downarrow$ is mean the amount of radiant energy in a narrow frequency interval that crosses unit area per unit time in the downward direction. I shall call $I_\downarrow$ the monochromatic intensity in the downward direction, or simply the downward intensity.[16] $I_\uparrow$ is defined similarly for the upward direction.

$I_\downarrow$ and $I_\uparrow$ change, in general, with the depth z into the medium because of absorption and scattering, which are specified by the (volumetric) absorption coefficient κ and the (volumetric) scattering coefficient β. These cannot be obtained within the framework of radiative transfer theory (a *macroscopic* theory); recourse must be had to *microscopic* theories. For a collection of independent identical scatterers the scattering coefficient β is simply the individual scattering cross section times the number density (number per unit volume). Thus $1/\beta$ has the dimensions of length and may be interpreted as the scattering mean free path (i.e., the average distance between scattering events). Similarly, κ is the absorption cross section times the number density and $1/\kappa$ is the absorption mean free path.

One further assumption will be made before proceeding: There are no time delays. That is, the dimensions of the medium are such that any change in the external illumination is felt instantaneously throughout it.

Now we apply conservation of radiant energy to Δz, first for the downward intensity:

$$I_\downarrow(z)+\beta\Delta z p_{\uparrow\downarrow} I_\uparrow(z+\Delta z) = \kappa\Delta z I_\downarrow(z)+\beta\Delta z p_{\downarrow\uparrow} I_\downarrow(z)+I_\downarrow(z+\Delta z)\,, \tag{1}$$

where $p_{\downarrow\uparrow}$ is the probability that a photon directed downward is scattered upward (and similarly for $p_{\uparrow\downarrow}$, $p_{\uparrow\uparrow}$, and $p_{\downarrow\downarrow}$). The terms on the left side of Eq. (1) are gains, while those on the right side are losses. If we divide both sides of Eq. (1) by Δz and take the limit as $\Delta z \to 0$, we obtain the following differential equation:

$$\frac{dI_\downarrow}{dz} = -\kappa I_\downarrow - \beta p_{\downarrow\uparrow} I_\downarrow + \beta p_{\uparrow\downarrow} I_\uparrow\,, \tag{2}$$

and similarly for the upward intensity:

$$\frac{dI_\uparrow}{dz} = \kappa I_\uparrow + \beta p_{\uparrow\downarrow} I_\uparrow - \beta p_{\downarrow\uparrow} I_\downarrow\,. \tag{3}$$

The sign reversal between Eqs. (2) and (3) occurs because the downward intensity is attenuated in the direction of increasing z, whereas the upward intensity is attenuated in the direction of decreasing z.

We take the medium to be isotropic, that is, $p_{\uparrow\downarrow}=p_{\downarrow\uparrow}$ and $p_{\downarrow\downarrow}=p_{\uparrow\uparrow}$. This is to be distinguished from *isotropic scattering* ($p_{\uparrow\downarrow}=p_{\uparrow\uparrow}$ and $p_{\downarrow\uparrow}=p_{\downarrow\downarrow}$) or an *isotropic radiation field* ($I_\downarrow=I_\uparrow$). As we shall see, isotropic scattering does not necessarily give rise to an isotropic radiation field, nor does anisotropic scattering necessarily give rise to an anisotropic radiation field. Examples of isotropic media are collections of spherical scatterers or randomly oriented nonspherical scatterers. A medium composed of nonspherical *oriented* scatterers is anisotropic.

In our model, photons must be scattered either forward or backward, which requires that

$$p_{\uparrow\downarrow}+p_{\uparrow\uparrow}=p_{\downarrow\uparrow}+p_{\downarrow\downarrow}=1\,. \tag{4}$$

The *asymmetry parameter g*, defined as the mean cosine of the scattering angle (which for us has only two values, 1 and -1)

$$g=(1)p_{\downarrow\downarrow}+(-1)p_{\downarrow\uparrow} \tag{5}$$

is a single number that specifies the degree of anisotropy of the scattering. It lies between 1 (strict forward scattering) and -1 (strict backward scattering); it is 0 for isotropic scattering. From Eqs. (4) and (5) it follows that the four probabilities in Eqs. (2) and (3) can be expressed in terms of g only:

$$p_{\uparrow\downarrow}=p_{\downarrow\uparrow}=(1-g)/2\,,\quad p_{\uparrow\uparrow}=p_{\downarrow\downarrow}=(1+g)/2\,. \tag{6}$$

If we divide Eqs. (2) and (3) by $\kappa+\beta$, transform the variable from *physical depth z* to *optical depth* τ defined by

$$\tau=\int_0^z(\kappa+\beta)\,dz\,, \tag{7}$$

and use Eq. (6), Eqs. (2) and (3) become

$$\frac{dI_\downarrow}{d\tau}=-I_\downarrow+\varpi_0\frac{1+g}{2}I_\downarrow+\varpi_0\frac{1-g}{2}I_\uparrow\,, \tag{8}$$

$$\frac{dI_\uparrow}{d\tau}=I_\uparrow-\varpi_0\frac{1+g}{2}I_\uparrow-\varpi_0\frac{1-g}{2}I_\downarrow\,, \tag{9}$$

where the *single-scattering albedo* $\overline{\omega}_0$ is defined as $\beta/(\kappa+\beta)$. If the medium is uniform, the optical depth, which is dimensionless, is $z(\kappa+\beta)$, and may be interpreted as the depth in units of total mean free path $1/(\kappa+\beta)$. Because the absorption and scattering coefficients depend on frequency, in general, so does the optical depth. The single-scattering albedo varies between 0 (no scattering) and 1 (no absorption). Both limits are idealizations never realized in practice.

Equations (8) and (9) are the two-stream equations of radiative transfer. Although we shall not do so, these equations can be extended to N streams, in which instance we obtain N coupled differential equations of the same form as Eqs. (8) or (9) but with $N+1$ terms on the right side; one term represents attenuation and the N remaining terms represent all the possible ways in which light is scattered into one direction from all other directions. In the limit as N goes to infinity, sums become integrals and the set of equations collapses into a single integro-differential equation.[1,2]

A more compact form of Eqs. (8) and (9) is obtained by first adding and then subtracting them:

$$\frac{d}{d\tau}(I_\downarrow - I_\uparrow) = -(1-\overline{\omega}_0)(I_\downarrow + I_\uparrow)\,, \qquad (10)$$

$$\frac{d}{d\tau}(I_\downarrow + I_\uparrow) = -(1-\overline{\omega}_0 g)(I_\downarrow - I_\uparrow)\,. \qquad (11)$$

Equations (8) and (9) [or, equivalently, Eqs. (10) and (11)] are consequences merely of conservation of energy applied to streams of photons constrained to only two directions. Equations like these were first obtained by Schuster,[10] although he included emission and restricted himself to isotropic scattering ($g=0$).

Now we shall try to obtain as much physical insight as possible from these simple equations by solving them subject to various boundary conditions, by interpreting the solutions, and by making a connection between them and observations.

III. CONSERVATIVE SCATTERING: NO ABSORPTION

No medium is strictly nonabsorbing. Nevertheless, we can sometimes ignore absorption and set the single-scattering albedo to unity without making great errors in calculated observable quantities (e.g., reflection of visible light by clouds). With this assumption Eqs. (10) and (11) have the simple solutions

$$I_\downarrow = D + C(1-\tau^*)\,, \quad I_\uparrow = D - C(1+\tau^*)\,, \qquad (12)$$

provided that g is independent of τ, where $\tau^* = (1-g)\tau$ is the *scaled* optical depth. The constants C and D are determined by conditions at the upper ($\tau=0$) and lower ($\tau=\overline{\tau}$) boundaries, where the (total) *optical thickness* of a medium with physical thickness h is

$$\overline{\tau} = \int_0^h (\kappa+\beta)\,dz\,. \qquad (13)$$

A. Equilibrium solution

Suppose that the medium is illuminated from above $[I_\downarrow(0) = I_0]$ and that a perfect reflector underlies it $[I_\uparrow(\overline{\tau}) = I_\downarrow(\overline{\tau})]$. For this case $C=0$ and $D=I_0$, so that the radiation field is uniform and isotropic (i.e., $I_\downarrow = I_\uparrow$ for all $\overline{\tau}_0$ and g). This is the *equilibrium solution*. To understand why this is so called, consider the medium to be enclosed by *two* perfect reflectors, above and below. The solution to Eq. (12) in this instance is $C=0$ and D is arbitrary. That is, if photons are introduced into the medium, they rattle around (none are absorbed or leak from the medium) until the intensity is everywhere uniform and isotropic.

B. Reflection and transmission

Suppose that photons that leak out of the lower boundary of the medium are not returned to it, either because there is nothing to scatter them back or they are absorbed [i.e., $I_\uparrow(\overline{\tau}) = 0$]. In this case, solution of Eq. (12) yields the following expressions for the *albedo*[17] (or *reflection coefficient*) R and *transmission coefficient* T:

$$R = I_\uparrow(0)/I_0 = \overline{\tau}^*/(2+\overline{\tau}^*)\,, \qquad (14)$$

$$T = I_\downarrow(\overline{\tau})/I_0 = 2/(2+\overline{\tau}^*)\,, \qquad (15)$$

which satisfy $R+T=1$.

Note that only the *scaled* optical thickness $\overline{\tau}^*$ determines these two observable properties of a (nonabsorbing) multiple-scattering medium. Thus two such media are optically similar if the product of $\overline{\tau}$ and $(1-g)$ is the same for both rather than $\overline{\tau}$ and g separately. Therefore, it is not possible to determine both $\overline{\tau}$ and g uniquely by measuring transmission and reflection.

As the optical thickness increases, R and T approach the limits 1 and 0, respectively. R will be within 1% of its asymptotic value if $\overline{\tau} > 200/(1-g)$, which provides a criterion for when a medium can be taken to be *optically thick* (i.e., effectively infinitely thick).

If $g=1$ (forward scattering only), then $R=0$ regardless of the optical thickness. But it does not then follow that R is necessarily 1 if $g=-1$ (backward scattering only).

We get more insight into Eq. (15) if we rewrite it using Eq. (6): $T = 1/(1+p_{\downarrow\uparrow}\overline{\tau})$. The physical interpretation of this is that photons are lost (strictly) to the downward stream only if they are scattered in the opposite direction.

For small (scaled) optical thickness we can expand Eq. (15):

$$T \simeq 1 - \overline{\tau}^*/2 \simeq \exp(-\overline{\tau}^*/2)\,, \quad (\overline{\tau}^* \ll 1)\,.$$

Thus the incident intensity is attenuated exponentially only if multiple scattering is negligible. Suppose that once a photon has been scattered in the backward direction, it is removed from the medium. In this instance, there is no multiple scattering, and the corresponding transmission coefficient $\exp(-\overline{\tau}^*/2)$ is always less than that given by Eq. (15). All else being equal, therefore, attenuation is *less* in a multiple-scattering medium than in a single-scattering medium, which is perhaps contrary to what one expects because of the word "multiple." This is so because photons scattered out of a particular direction can find their way back into that direction by being scattered again one or more times.

C. Reflection and transmission by clouds

Multiple-scattering media such as clouds have high albedos (at visible wavelengths) because incident photons reemerge after having been scattered many times by particles that are only weakly absorbing. One cannot explain observations of clouds on the basis of single-scattering arguments.

Nevertheless, one encounters frequently the statement that clouds are white because they are composed of nonselective scatterers. It is true that cloud droplets are so large compared with the wavelengths of visible light that their scattering cross sections are nearly independent of wavelength. This is a *sufficient* condition for the whiteness of clouds, but it is not *necessary*. The converse is not true: A collection of selective scatterers is not necessarily colored. For example, milk is a suspension of small particles that scatter blue light more than red. You can demonstrate this by adding a few drops of milk to water and illuminating the resulting suspension with a collimated beam of white light. The scattered light will be bluish and that transmitted will be reddish. Yet a glass of milk is white. We can understand why by differentiating Eq. (14) with respect to the wavelength λ:

$$\frac{dR}{d\lambda} = \frac{2}{(2+\overline{\tau}^*)^2}\frac{d\overline{\tau}^*}{d\lambda}. \tag{16}$$

If the optical thickness $\overline{\tau}^*$ (which is proportional to the scattering coefficient β) is independent of wavelength then so is R. But the converse is not true. For sufficiently large optical thickness, R is independent of wavelength regardless of the wavelength dependence of $\overline{\tau}^*$.

To proceed further we need to estimate the optical thickness of clouds. For simplicity, let us assume that they are uniform and are composed of droplets all of which have the same radius a. The optical thickness of a suspension of N identical particles per unit volume is

$$\overline{\tau} = NC_{\text{ext}}\, h = f(C_{\text{ext}}/v)\, h\,, \tag{17}$$

where the extinction cross section C_{ext} is the sum of absorption and scattering cross sections (i.e., the particle's effective cross-sectional areas for removal of photons from a beam by absorption and scattering), v is the volume of a particle, h is the thickness of the suspension, and $f = Nv$ is the fraction of the total suspension volume occupied by the particles. For spheres much larger than the wavelength of the light illuminating them, C_{ext} is approximately twice the geometrical cross section πa^2 (see, e.g., Ref. 18, p. 107). Clouds are quite tenuous: f is typically around 3×10^{-7}, corresponding to a liquid water content of 0.3 g/m^{-3} (Ref. 19, p. 15). Cloud droplets are distributed in size (Ref. 19, pp. 13 and 14), but a diameter of 10 μm is representative. With these assumptions, the *approximate* optical thickness of a cloud is $\overline{\tau} = 100\text{h}$, where h is in km. But it is the scaled optical thickness that appears in Eqs. (14) and (15), so we need to estimate the asymmetry parameter g. From the calculations for water droplets tabulated by Irvine and Pollack,[20] it is evident that $g = 0.85$ is a representative value for cloud droplets at visible wavelengths, that is, scattering of light by such droplets is highly peaked in the forward direction. Thus we obtain $\overline{\tau}^* \simeq 14h$ as a rough estimate for the scaled optical thickness of a cloud of physical thickness h. It follows from this result and Eq. (16) that the albedo of clouds thicker than aboout 1 km would be nearly independent of wavelength regardless of the wavelength dependence of scattering by the individual droplets.

The preceding statements may seem, at first glance, to be incompatible with the colors seen often in thin clouds or at the edges of thick clouds when looking toward the sun. Such colors, called iridescence, are consequences of single scattering. Although *total* scattering by a cloud droplet is nearly independent of wavelength, the angular distribution of the scattered light (i.e., the *differential* scattering cross section) is not. For more on iridescent clouds see Refs. 21–24.

It is rare for clouds overhead to be so thick that day becomes night. From Eq. (15) and our estimate for the optical thickness of clouds it follows that they would have to be more than 15-km thick to transmit less than 1% of the light incident on them. This would require the atmosphere to be filled uniformly with clouds from the surface to the tropopause.

D. Reflection by snow

Snow on the ground is one common example of a multiple-scattering medium that is often sufficiently deep for its albedo to be close to the asymptotic value. Ice grains in such snow are not necessarily spherical, although they are rarely the spatial dendrites favored by painters of Christmas cards (see Ref. 25 for a good discussion of how the shapes of snowflakes change with time after they settle). Like cloud droplets, these ice grains are nonselective scatterers of visible light. The volume fraction f of snow on the ground varies, but 0.3 is typical. The extinction cross section of any large particle is twice its geometrical cross section projected onto the beam illuminating it (Ref. 18, p. 107). Thus the extinction cross section of an ice grain is proportional to the square of a characteristic linear dimension d; its volume v is proportional to the cube of d. For ice grains in snow a representative value for d is about 1 mm. Because they are much larger than cloud droplets, scattering by the grains is more sharply peaked in the forward direction (i.e., g is closer to 1). Let us therefore take g to be 0.93, a reasonable estimate. Thus we estimate the (scaled) optical thickness $\overline{\tau}^*$ of a snowpack h meters deep to be $200h$ (for finer grained, denser snow the coefficient of h will be even greater). According to this estimate, snow about 1-m deep is optically thick (i.e., its albedo is within 1% of the asymptotic value). Absorption has been neglected; to include it would reduce the depth snow must be for it to be considered optically thick. In Sec. IV the observable consequences of not neglecting absorption by ice grains in snow will be discussed.

It may be inconvenient to tunnel deep into snow to verify my assertion about the optical thickness of snow. Observations are made more comfortably above snow. Suppose that a perfect reflector (i.e., a white surface) underlies the snow, in which case $R = 1$. Equation (14) applies to snow over a perfectly black surface. The difference between these two extreme albedos, which I shall denote as R_w and R_b, is

$$R_w - R_b = 2/(2+\overline{\tau}^*)\,.$$

Snow therefore does not have to be very deep, perhaps a few tens of centimeters, before it is not possible to tell what underlies it, which is often observable. Another observation you can make is of the comparative brightness of snow and clouds under similar illumination. Because deep snow is optically thicker than clouds, snow is usually brigher.

E. Diffuse radiation: Cloud light

Few of us are in the habit of staring directly at the sun, even when it is partially obscured by clouds. More often, we see *diffuse* radiation, that is, radiation that has been scattered (e.g., cloud light or skylight). If there is no illumination from below, the upward intensity $I_\uparrow$ is necessarily diffuse. To emphasize this, I shall henceforth denote this quantity as $D_\uparrow$. The downward intensity $I_\downarrow$ is the sum of two components, an unscattered component $I_\downarrow^u$ and a diffuse component $D_\downarrow$. We can determine these quantities by rewriting Eq. (2) as follows:

$$\frac{dI_\downarrow}{d\tau} + I_\downarrow = p_{\downarrow\uparrow} I_\downarrow + p_{\uparrow\downarrow} I_\uparrow .$$

The terms on the right side of this equation involve scattering. Thus the *unscattered* intensity satisfies the homogeneous equation

$$\frac{dI_\downarrow^u}{d\tau} + I_\downarrow^u = 0 ,$$

which has the solution $I_\downarrow^u = I_0 \exp(-\tau)$ for the boundary condition $I_\downarrow^u(0) = I_\downarrow(0) = I_0$. Let us take what underlies the medium to be perfectly black [i.e., $I_\uparrow(\overline{\tau}) = 0$]. At any arbitrary optical depth τ into the medium, the two diffuse intensities are

$$D_\uparrow / I_0 = [\tfrac{1}{2}(1-g)(\overline{\tau}-\tau)]/[1+\tfrac{1}{2}(1-g)\overline{\tau}] , \quad (18)$$

$$D_\downarrow / I_0 = \{[1+\tfrac{1}{2}(1-g)(\overline{\tau}-\tau)]/[1+\tfrac{1}{2}(1-g)\overline{\tau}]\} - \exp(-\tau) . \quad (19)$$

The upward intensity vanishes at the lower boundary, and has its greatest value at the upper boundary. The downward diffuse intensity vanishes at the upper boundary, increases to a maximum, and then decreases toward the lower boundary. It follows from inspection of Eqs. (18) and (19) that the two diffuse intensities are *approximately* equal for optical depths satisfying

$$2/(1-g) < \tau < \overline{\tau} - [2/(1-g)] . \quad (20)$$

This result has a simple physical interpretation, which follows from Eq. (6) and the definition of optical depth. The diffuse radiation field is highly anisotropic near boundaries, but because of scattering it tends toward isotropy as we move away from them. The quantity $\tau(1-g)/2$, which is (approximately) the fraction of photons that have been reversed in direction in traversing an optical depth τ, is 1 for $\tau = 2/(1-g)$. Thus at optical depths greater than $2/(1-g)$ from either boundary the diffuse radiation field is nearly isotropic. This isotropy of diffuse radiation deep within a multiple-scattering medium can be observed from an airplane descending through clouds. After entering the cloud, you soon will have no visual clues to tell up from down, but eventually you will see the upward intensity begin to decrease (unless you are flying over snow); this is the signal that your airplane is about to leave the cloud.

Now suppose that we are on the ground looking at clouds overhead.[26] The diffuse downward intensity $D_\downarrow(\overline{\tau})$, which from Eq. (19) is

$$D_\downarrow(\overline{\tau})/I_0 = \{1/[1+\tfrac{1}{2}(1-g)\overline{\tau}]\} - \exp(-\overline{\tau}) , \quad (21)$$

rises steeply with increasing optical thickness $\overline{\tau}$ beginning at $\overline{\tau} = 0$ to a maximum at $\overline{\tau} \simeq \ln[2/(1-g)]$, and then decreases more gradually with a further increase in $\overline{\tau}$ (Fig. 2). This has observable consequences. A thin cloud layer, say from a few tens to a few hundreds of meters thick, is brighter than the clear sky. You can sometimes see this when the sky is covered with thin clouds. If there are occasional breaks, you can compare the brightness of the clear and cloudy skies. Very thick clouds result in gloomy days, when the sky is less bright than on clear days.

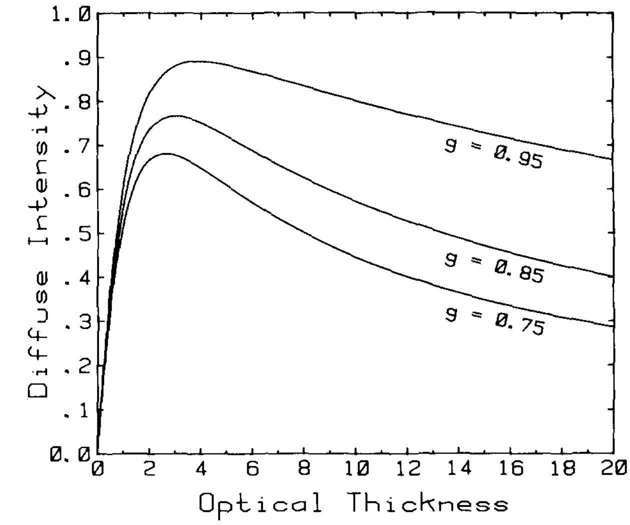

Fig. 2. Diffuse downward intensity of radiation below a nonabsorbing medium of finite optical thickness illuminated from above. g is the mean cosine of the scattering angle.

F. Diffuse radiation: Skylight

Let us now consider what we see when looking upward[26] on a very clear day. Even if the atmosphere were completely free of particles, we would still see a blue sky because of molecular scattering. Indeed, particles decrease the spectral purity of skylight.[27] Because the number density of molecules decreases with height, the molecular scattering coefficient is not uniform. But since this decrease is (approximately) exponential with a *scale height* $H = 8.4$ km, the optical thickness along a radial path from the surface to infinity is the same as that for an atmosphere extending from the surface to H with a constant density equal to the sea level value:

$$\beta_0 H = \int_0^\infty \beta_0 \exp(-z/H)\, dz ,$$

where β_0 is the sea level molecular scattering coefficient. The optical thickness shown in Fig. 3 was obtained from values of β_0 tabulated by Penndorf.[28]

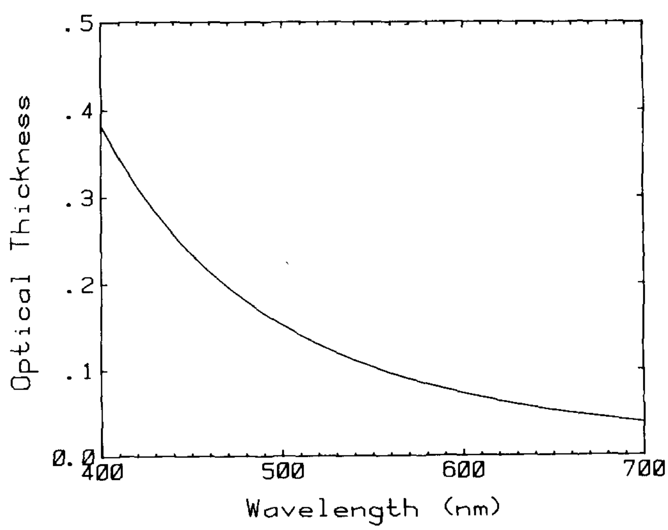

Fig. 3. Molecular optical thickness of Earth's atmosphere along a vertical path from the surface to infinity.

Molecular scattering of light is not isotropic, but it is symmetric about a scattering angle of 90°. Thus we may take $g = 0$ in Eq. (21) to obtain an approximate expression for the intensity of skylight overhead:

$$D_{\downarrow}/I_0 = [1/(1 + \tfrac{1}{2}\overline{\tau})] - \exp(-\overline{\tau}) \qquad (22)$$

for an atmosphere overlying black ground. If the ground is white (i.e., perfectly reflecting), the skylight intensity is

$$D_{\downarrow}/I_0 = 1 - \exp(-\overline{\tau}) \,. \qquad (23)$$

Since $\overline{\tau}$ is small (Fig. 3), we can expand Eqs. (22) and (23) to obtain

$$D_{\downarrow}^{b}/I_0 \simeq \tfrac{1}{2}\overline{\tau}, \quad D_{\downarrow}^{w}/I_0 \simeq \overline{\tau}, \qquad (24)$$

where the superscripts indicate the nature of the ground. In both instances we obtain a blue sky (provided we take into account the response of the human eye; for more about this see Ref. 27), but the sky brightness over black ground is about half that over white ground. This makes sense: Both direct sunlight and reflected groundlight illuminate the scatterers.

Only because $\overline{\tau}$ is small do we have a blue sky. If our atmosphere were much thicker, the sky would appear quite different. This is shown in Figs. 4 and 5 (see also Refs. 27 and 29). Skylight intensity is shown as a function of wavelength for both white and black ground. The optical thicknesses are for Earth's actual atmosphere, and for ones with ten and 30 times as many molecules. Note that increasing the number of molecules, which scatter very nearly according to the inverse fourth power of the wavelength, does not make the sky bluer. On the contrary, the intensity increases at the expense of spectral purity. Here is yet another place where multiple scattering must be invoked. In the absence of multiple scattering, an increase in optical thickness would make the sky brigher with *no* change in its spectrum.

Now that we have sharpened our physical intuition, let us be bold and briefly step outside of the one-dimensional domain imposed by the two-stream model. Suppose that the sun is overhead on an extraordinarily clear day, and that we have an unimpeded view of the horizon. What does the sky look like in different directions? It is not, as predicted by single scattering, uniform in color. The sky is bluest near the zenith; it is brightest, but of low spectral purity, near the horizon. The optical thickness (called the *normal*

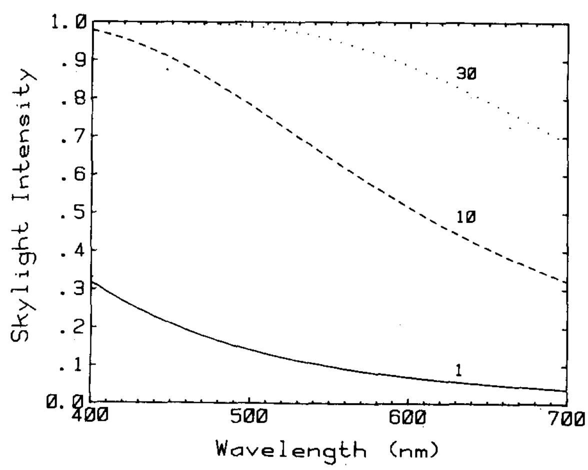

Fig. 4. Skylight intensity (overhead sky) over white ground. The numbers denote the optical thicknesses of hypothetical pure molecular atmospheres relative to the actual value for Earth's atmosphere.

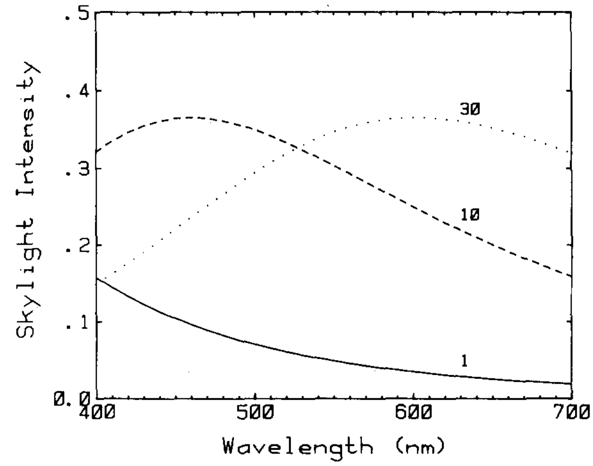

Fig. 5. Skylight intensity (overhead sky) over black ground. The numbers denote the optical thicknesses of hypothetical pure molecular atmospheres relative to the actual value for Earth's atmosphere. Note that the vertical scale is one-half that in Fig. 4.

optical thickness and denoted by $\overline{\tau}_n$) in Eq. (23) is that of a vertical path through the atmosphere, so it applies strictly to the zenith sky. But let us assume, on physical grounds, that this equation is valid even if $\overline{\tau}$ is that for a slant path through the atmosphere. Except for directions very close to the horizon, the optical thickness of a slant path is $\overline{\tau}_n/\cos\theta$, where θ is the angle measured from the zenith. With this assumption, the diffuse intensity in the direction θ is

$$D_{\downarrow}/I_0 = 1 - \exp(-\overline{\tau}_n/\cos\theta) \,. \qquad (25)$$

Equation (25), approximate to be sure, is in accord with observations of the color and brightness variation of clear skies. To further test its validity we can compare it with the exact calculations of Coulson *et al.*[30] for a pure molecular atmosphere (over a flat Earth). This is done in Fig. 6, from which it is evident that the simple theory [Eq. (25)] is adequate to explain the variation in brightness and color of the sky. The best place to observe a sky predicted by Eq. (25), which applies to a sky over a highly reflecting surface, is from an airplane flying over clouds. At the altitudes where commercial aircraft fly, there are not many particles: The scale height for atmospheric particles is a few kilometers. So we can be reasonably sure that what we see—dark blues near the zenith and a whitish horizon—is a property of an atmosphere in which scattering by molecules is predominant.

Before we consider absorption, there is another observation to which the preceding analysis may be applied: the appearance of distant objects. One receives not only light from such objects, but *airlight*—light scattered by everything along the line of sight—as well. If the object is black, then *all* the light one receives is airlight. Its intensity is proportional to $1 - \exp(-\tau)$, where τ is the optical thickness of the path from observer to object (see Ref. 31 for more details). If the scatterers are mostly molecules and very small particles, the optical thickness will be inversely proportional to the inverse fourth power of the wavelength. For small optical thicknesses ($\ll 1$) the airlight intensity is proportional to τ and hence is bluish. But as τ is increased, the airlight intensity increases in magnitude while at the same time becomes less dependent on wavelength. This can be observed in a series of parallel mountain ridges, one be-

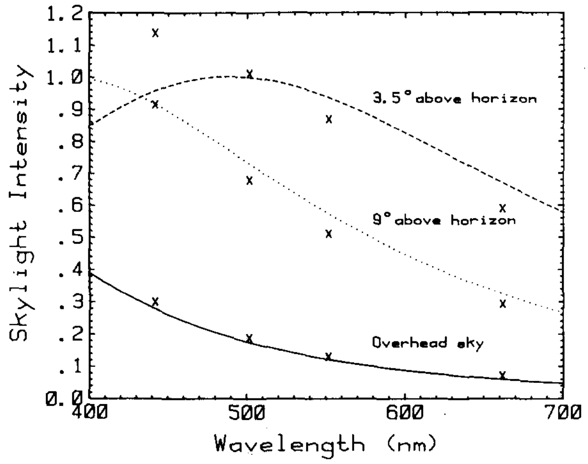

Fig. 6. Skylight intensity for a pure molecular atmosphere. The curves were calculated using the simple two-stream theory of this paper. The ground albedo 0.8 is sufficiently close to 1 that Eq. (25) is a good approximation, although the calculations were done using an expression for the diffuse downward intensity over ground with arbitrary albedo [this expression is not given, but it follows readily from Eq. (12)]. Here (——) is for the sky directly overhead; (- - -) is for that about 3.5° above the horizon; (···) is for that about 9 degrees; Xs show the detailed calculations of Coulson *et al.*[30] for the principal plane (determined by the direction of the incident sunlight and the normal to the ground) with the sun 23 degrees from the zenith. All intensities have been normalized by that at 500 nm for the near-horizon sky.

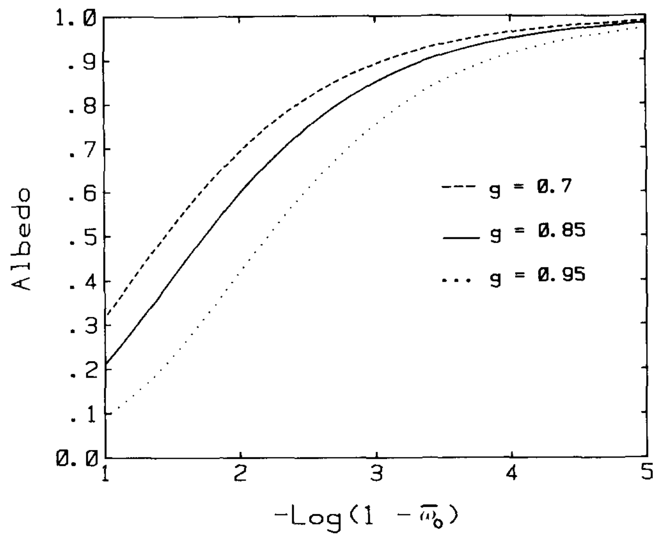

Fig. 7. Albedo of an infinitely thick medium as a function of single-scattering allbedo $\bar{\omega}_0$ for several asymmetry parameters: (- - -) $g = 0.7$; (——) $g = 0.85$; (···) $g = 0.95$.

hind the other, covered with dark vegetation. The closest ridges will have their natural color; those somewhat farther will be bluish; the farthest ridges fade into the whitish horizon sky. To explain this common observation, one must invoke multiple scattering.

IV. MULTIPLE SCATTERING IN ABSORBING MEDIA

Let us now consider the more general case of multiple scattering in an absorbing medium ($\kappa > 0$ hence $\bar{\omega}_0 < 1$). By differentiating Eq. (10) [Eq. (11)] with respect to τ, substituting Eq. (11) [Eq. (10)] in the result, then adding and subtracting the two equations obtained, it follows that

$$\frac{d^2F}{d\tau^2} = K^2F,$$

$$K = \sqrt{(1 - \bar{\omega}_0)(1 - \bar{\omega}_0 g)}\,, \tag{26}$$

where F is either the sum or difference of the two intensities. Solutions to Eq. (26) are linear combinations of the two exponential functions $\exp(\pm K\tau)$. Since the intended applications are to optically thick media, we may take the medium to be infinitely thick. This will simplify the mathematics because the coefficients of $\exp(K\tau)$ must vanish in order to ensure finite intensities. By using Eqs. (10) or (11) and the boundary condition $I_\downarrow(0) = I_0$, a bit of algebra gives the downward and upward intensities:

$$I_\downarrow = I_0 \exp(-K\tau)\,, \quad I_\uparrow = I_0 R \exp(-K\tau)\,, \tag{27}$$

where the albedo $R = I_\uparrow(0)/I_0$ is

$$R = (\sqrt{1-\bar{\omega}_0 g} - \sqrt{1-\bar{\omega}_0})/(\sqrt{1-\bar{\omega}_0 g} + \sqrt{1-\bar{\omega}_0})\,. \tag{28}$$

As a check on the correctness of Eq. (28), we note that $R = 1$ when $\bar{\omega}_0 = 1$. Also, for $\bar{\omega}_0 \neq 1$, $R = 0$ for $g = 1$: With no mechanism for redirecting incident photons, they can never reemerge from the medium. Note that the derivative of R with respect to $\bar{\omega}_0$ at $\bar{\omega}_0 = 1$ is infinite. This is a quantitative statement of what can be stated as an aphorism: In a multiple-scattering medium, a little bit of absorption goes a long way. The curves in Fig. 7 show the albedo R as a function of single-scattering albedo for a few values of g. We interpret $1 - \bar{\omega}_0$ as the probability that a photon is absorbed in a *single* interaction with a scatterer; $1 - R$ is the probability that it will be absorbed in *many* such interactions. If $1 - \bar{\omega}_0 = 0.001$, for example, then for $g = 0.9$, $1 - R = 0.18$. Thus there is an enormous amplification of absorption—nearly a factor of 200 in this example—in going from single to multiple scattering. For an incident downward photon to find its way into the upward intensity outside the medium, it must, on average, be scattered many times; the greater g is, the greater the number of scatterings (in the extreme case $g = 1$, incident photons never escape). In each scattering, a photon has a small chance of being absorbed, but in many scatterings its cumulative chance of being absorbed increases markedly.

In the context of multiple scattering, a single-scattering albedo of 0.9 (or even 0.99), which at first glance seems high, is in fact quite low, especially if the scatterers are larger than the wavelength and hence have large values of g (see, e.g., Ref. 32, pp. 385–388; Ref. 33, p. 183). This has consequences for the commonly made statement that temperatures are higher on cloudy nights than on clear nights because infrared radiation emitted by the Earth is "reflected" by the clouds. Although the observation is correct, this explanation is not. The single-scattering albedo of cloud droplets is quite high at visible and near-visible wavelengths, greater than 0.99999. But in the infrared it plunges because of the infrared absorption bands of water. The Earth emits radiation of all wavelengths, but most of it is between about 8 and 12 μm. At these wavelengths, the single-scattering albedo of typical cloud droplets is at most about 0.8 (see Ref. 20, Table V), which seems high, but is in fact quite low (see Fig. 7). As a consequence, clouds are nearly black to radiation emitted by the Earth. If clouds elevate nighttime temperatures, it is not because they reflect infrared radiation, but rather that they emit more of it than the clear night sky does.

A. Darkening of sand upon wetting

With Eq. (28) and Fig. 7 we can explain another common observation: the darkening of sand upon wetting. Sand is usually an optically thick multiple-scattering medium (i.e., it is so thick that the addition of another layer does not sensibly change its appearance). When dry sand is wetted, it becomes noticeably darker. At visible wavelengths, water is very weakly absorbing, so increased absorption by water is certainly not the reason for this. Grains in dry sand are surrounded by air. But when wet (strictly speaking, when the sand is saturated), they are surrounded by water. In going from air to water (or other liquids), the grains scatter more toward the forward direction. Indeed, when the refractive index of the grain is exactly that of the surrounding liquid, scattering is entirely in the forward direction. Thus when a large grain is surrounded by a liquid instead of by air, its asymmetry parameter g increases. As the average degree of forwardness of scattering increases, incident photons have to be scattered more times before reemerging from the sand and are therefore exposed to a greater probability of being absorbed. For more details about this explanation of the darkening of sand and experimental verification of it see Ref. 34.

While we are on the subject of sand, we can use Eq. (28) to explain another observation. The asymmetry parameter of very large particles does not depend strongly on their size or on the wavelength. If the particles are weakly absorbing, their absorption cross section is proportional to the product of the absorption coefficient α of the bulk parent material and the particle volume (i.e., the cube of a characteristic linear dimension d). But the extinction cross section (sum of absorption and scattering cross sections) is proportional to the square of d; hence $1-\overline{\omega}_0$ is proportional to αd. If we expand Eq. (28) in powers of $\sqrt{\alpha d}$ and retain only the first term, we obtain

$$R \simeq 1 - C\sqrt{\alpha d}\ , \qquad (29)$$

where C is a wavelength-independent factor that depends on, among other things, the asymmetry parameter. Thus the smaller the grains, the closer the albedo is to 1. This can be observed on a beach with sands that have been size segregated by wind or water: The coarser sand is darker (see Ref. 35 for a photograph). If there is not a beach nearby you can verify the qualitative correctness of Eq. (29) by smashing colored beer bottles into small bits. According to Eq. (29), the smaller the grains, the higher the albedo. So if you smash the bottles into very small bits, the resulting heap of powdered glass will have a high albedo which will not be noticeably wavelength dependent. But by selecting sufficiently large glass particles (by sieving, for example), the heap of powdered glass will be less bright and may even display the color (although of lower spectral purity) of the parent bottles.

B. Colors in snow

I mentioned in Sec. III D that snow is another common multiple-scattering medium (for a good review of the optical properties of snow, see Ref. 36). In a recent popular article,[37] I found the following statement: "Snow is not white; it simply reflects the light, which is—usually—white." Rather than comment on this explanation, I shall try to give a more satisfactory one. Ice grains in snow scatter predominantly near the forward direction (i.e., g is close to 1). Incident photons that reemerge from a snowpack must therefore have been scattered many times. If they are to survive many scatterings, the probability of absorption $(1-\overline{\omega}_0)$ must be small, which it is at visible wavelengths. Visible light must be transmitted through many meters of pure homogeneous ice to be appreciably absorbed.[38,39] Ice is intrinsically blue: It absorbs red light much more than blue light. But $1/\alpha$ for ice is of order meters over the visible spectrum, whereas grain sizes are of order millimeters. Thus the product αd for ice grains in snow is very small, and as a consequence [see Eq. (29)] the albedo of snow is high and does not vary appreciably over the visible spectrum.

But this does not mean that snow does not exhibit colors. Deep blues can be seen in crevasses, in ice caves, and even in holes in ordinary snow (see Ref. 40 and the cover of Ref. 41 for photographs of blue holes in snow). Raman[42] attributed this blueness to scattering by molecules, an incorrect explanation that is still extant,[43] although an essentially correct explanation was given in 1886 by Walthère Spring.[44] Raman's[42] arguments apply only to ice of a transparency much greater than that usually found in nature. Molecular scattering in snow is overwhelmed by scattering by ice grains. Moreover, if Raman's explanation were correct, light transmitted in snow would be red, not blue. If K [see Eqs. (27) and (28)] were independent of wavelength, and τ inversely proportional to the fourth power of the wavelength, then the transmitted intensity would get redder with increasing depth into the snow. This is contrary to what is observed.

The optical thickness of snow is to good approximation independent of wavelength, from visible to infrared, because the extinction cross section of ice grains is nearly constant in this range. This statement seems to contradict the observation that sunlight transmitted into snow is blue. Here is a good example of the great difference between single and multiple scattering. For small optical thicknesses, multiple scattering is negligible, and all scattered photons are lost from an incident beam. Of course, photons can also be absorbed. It is the *sum* of scattering and absorption that is independent of wavelength. Multiple scattering in snow is not negligible. Attenuation in snow will therefore be less than it would otherwise be (i.e., with no multiple scattering) because photons scattered out of the direction of a beam can reappear in this direction. But it is only those photons that are *not* absorbed that do so. Visible light at the short wavelength end of the spectrum is less likely to be absorbed by an ice grain than light at the long wavelength end. Hence, it is only because of nonselective multiple scattering in a selectively absorbing medium that we see blue light in snow holes. Moreover, the depth to which white light must be transmitted in snow to become blue is considerably less, a factor of about 50, than it would have to be transmitted in homogeneous ice: Multiple scattering in snow increases the total path length a photon travels through ice before reaching a given depth. Although absorption by liquid water over the visible spectrum is quite similar to that by ice, we do not see vivid blue light beneath clouds because their optical thicknesses are smaller than those of deep snow and because cloud droplets are much smaller than ice grains.

As a final example, to inspire confidence in the simple-

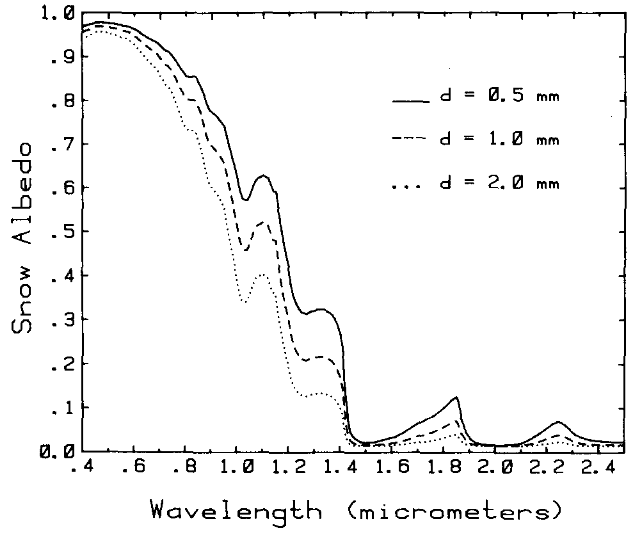

Fig. 8. Albedo of optically thick snow, calculated using the simple two-stream theory of radiative transfer, for various grain diameters: (——) $d = 0.5$ mm; (- - -) $d = 1.0$ mm; (···) $d = 2.0$ mm.

two stream theory, I have used Eq. (28) to compute the albedo of snow over most of the solar spectrum. The grains are taken to be spheres, all with the same diameter d, and with an asymmetry parameter of 0.94. For the single-scattering albedo, I have used the approximation $\overline{\omega}_0 = 1 - Q_{abs}/2$, where the absorption efficiency Q_{abs} (absorption cross section normalized by geometrical cross section) is obtained from Ref. 45. The refractive index of ice is taken to be 1.3 over the solar spectrum; its absorption coefficient is obtained from the compilation by Warren.[39] The results are shown in Fig. 8. Despite all these simplifications, the calculated albedo agrees well with that measured by O'Brien and Munis[46] (see their Fig. 2). All the wiggles and bumps, corresponding to the absorption bands of bulk ice, are in their proper positions. Even the magnitude of the albedo is approximately correct, although theory tends to overestimate the visible albedo. The reason for this is probably contamination.[47] Absorption of visible light by carbonaceous materials (lumped under the heading of "soot") is more than a million times greater than absorption by ice. As a consequence, a few parts per million of soot in snow can markedly reduce its visible albedo. In the infrared, however, ice is so strongly absorbing that an absorbing contaminant in snow does not change its albedo.

Figure 8 illustrates a statement I make to get my students scratching their heads: Snow is the whitest natural substance on our planet; it is also the blackest. At visible wavelengths, the albedo of clean, fine-grained Antarctic snow[48] is as high as 0.97. But in the middle infrared, the albedo of snow is so low that it is nearly a perfect blackbody.

V. CONCLUDING REMARKS

Just as it is not necessary to begin with the Maxwell equations to understand coherent scattering (in the guise of reflection and refraction by homogeneous, optically smooth media), it is not necessary to begin with the exact equation of radiative transfer to understand incoherent scattering. I have approached multiple (incoherent) scattering by way of a simple two-stream theory. Its major defect is that it suppresses the full directionality of the radiation field: 2π Sr are collapsed into a single direction. Despite this simplification, the results of the two-stream theory are applicable to a great many common observations, ones which cannot be understood by single-scattering arguments. My approach has been to first derive mathematical expressions, then apply them to observations, and finally to give a physical interpretation. Much of the terminology introduced in this simple context—single-scattering albedo, asymmetry parameter, optical depth, optical thickness, etc.—appears in more advanced treatises, so the way has been paved for further study by those whose appetites have been whetted. Some of the ideas in this paper, and simple ways to demonstrate them, are discussed at an elementary level (i.e., without mathematics) elsewhere.[35,49]

ACKNOWLEDGMENTS

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[17] In general, albedo and reflection coefficient (also called reflectance and reflection function) refer to different quantities: Albedo is the fraction of the light incident on the medium that is reflected in the entire backward hemisphere, whereas reflection coefficient is the fraction reflected

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The rattle in the cradle

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The history of science has known for quite some time that Newton's very first concept of centripetal force is derived from a simple model of a particle bouncing elastically in a polygonal path from a circular constraint. Fully described by Isaac Newton himself in his "Principia" 300 years ago, the model does not seem to have caused any response in physics courses or textbooks.

I. INTRODUCTION

Most teachers are probably aware of the difficulty of convincing students that the motion of a particle along any curvilinear path is accelerated and in particular that the revolution of a body at uniform speed along a circle is also accompanied with acceleration, notwithstanding the fact that this should be clear already from the First Law of Motion. The correct introduction of vectors and vector derivatives, although indispensable, does not seem to stir the alchemy of everybody's mind in the same way. It appears that either "uniform speed" is translated subconciously into "constant velocity," or, which is more probable, that students retain firmly their everyday concept of acceleration. After all, unlike tangential, normal acceleration is not acceleration *at all* in the everyday sense of the word.

Having in mind this difficulty, and also more subtle ones, which arise at subsequent stages of the teaching process, i.e., when motion is analyzed in accelerated and in particular in rotating coordinate systems, or when the Mach and Einstein point of view on relative rotations is to be brought out, it is suprising to see that university textbooks reflect so little of those explanations of circular motion that were historically the first to be given.

Even today, facing the task of persuading a young person that motion on a circle is accompanied with acceleration, it is best not to rely on the properties of vector derivatives, but on the example given in Isaac Newton's "Principia,"[1,2] in the text following the Fifth Definition. This well-known example starts with a shot fired horizontally from a gun at the top of a high mountain, which is a case of accelerated motion, and then proceeds by increasing the initial velocity in the absence of air, until finally the projectile is given sufficient horizontal velocity to become a satellite of the Earth (or even leave its field of gravity forever). The fact that the passage from free fall and projectile motion to circular motion is effected by continuous variation of one *initial* condition, the horizontal velocity of the projectile,

