Two-Stream Approximations to Radiative Transfer in Planetary Atmospheres: A Unified Description of Existing Methods and a New Improvement

W. E. MEADOR AND W. R. WEAVER

NASA, Langley Research Center, Hampton, VA 23665

ABSTRACT

Existing two-stream approximations to radiative transfer theory for particulate media are shown to be represented by identical forms of coupled differential equations if the intensity is replaced by integrals of the intensity over hemispheres. One set of solutions thus suffices for all methods and provides convenient analytical comparisons. The equations also suggest modifications of the standard techniques so as to duplicate exact solutions for thin atmospheres and thus permit accurate determinations of the effects of typical aerosol layers. Numerical results for the plane albedos of plane-parallel atmospheres (single-scattering albedo = 0.8, 1.0; optical thickness = 0.25, 1, 4, 16; Henyey-Greenstein phase function with asymmetry factor 0.75) are given for conventional and modified Eddington approximations, conventional and modified two-point quadrature schemes, the hemispheric-constant method and the delta-function method, all for comparison with accurate discrete-ordinate solutions. A new two-stream approximation is introduced that reduces to the modified Eddington approximation in the limit of isotropic phase functions and to the exact solution in the limit of extreme anisotropic scattering. Comparisons of plane albedos and transmittances show the new method to be generally superior over a wide range of atmospheric conditions (including cloud and aerosol layers), especially in the case of nonconservative scattering.

1. Introduction

Two-stream methods have been widely used for many years in providing rapid approximate answers to problems of radiative transfer in particulate materials. They avoid the complex and lengthy computer procedures necessary for numerical solutions of the radiative transfer equation, while yielding closed-form analytical results that are relatively easy to interpret and that often represent adequately some of the most important features of multiple-scattering processes. Although many of the more recent two-stream studies have centered on such atmospheric problems as the effects on planetary albedos of haze and clouds (cf. Sagan and Pollack, 1967; Liou, 1973, 1974; Lyzenga, 1973), solar irradiance through inhomogeneous turbid atmospheres (Shettle and Weinman, 1970), global climatic modeling (Weare and Snell, 1974; Temkin et al., 1975) and the climatic effects of aerosols (Rasool and Schneider, 1971; Coakley and Chýlek, 1975), most of the fundamental concepts were formulated much earlier for different purposes and resulted in successful industrial applications to radiative transfer in pigmented films (e.g., paint), opal glass, plastics, paper, rubber and textiles (cf., Kortüm, 1969).

One of the difficulties inherent in any comprehensive treatment of two-stream approximations is the number and variety of methods that have been used.

While some methods are distinctly different from others, some are only slightly different but have never been correlated apparently because of the lack of a basic theoretical framework within which comparisons can be made. The present paper begins with the development of such a framework and shows that existing two-stream approximations can be represented by identical forms of coupled differential equations if the radiation intensity is replaced by integrals of the intensity over hemispheres. These integrals are directly related to planetary albedos and to the transmission of radiation through atmospheres and can also be used for determining such atmospheric properties as the heating caused by aerosol layers. Since two-stream methods are reliable only for integrated quantities, as opposed to angular-dependent intensities, the introduction of integrals at an early stage is not a real limitation on the applicability of results.

Important features of the consolidation of methods include the direct analytical comparisons that it provides, the inclusion of some methods that were previously thought to be conceptually or mathematically different, and the existence of standard forms of solution that pertain to any given method with the appropriate choice of parameters. Recent review papers by Irvine and Lenoble (1974) and by Irvine (1975), for example, have discussed the differences between certain selected approximations

rather than the perhaps more important similarities. Another feature of the consolidation is the suggestion of modifications to standard two-stream techniques so as to duplicate exact solutions for thin atmospheres. The new equations should be particularly appropriate for determining the effects of typical aerosol layers and, in some cases at least, are shown to be almost as applicable to optically thick media such as clouds.

Numerical results for the plane albedos of plane-parallel atmospheres are given over a wide range of variables for conventional and modified Eddington approximations, conventional and modified two-point quadrature schemes, the hemispheric constant method and the delta-function method, all for comparison with accurate discrete-ordinate solutions. A new two-stream approximation is introduced that reduces to the modified Eddington approximation in the limit of isotropic phase functions and to the exact solution in the limit of extreme anisotropic scattering. Comparisons of plane albedos and transmittances show this method to be generally superior over large ranges of atmospheric conditions, especially in the case of nonconservative scattering.

2. Basic theory

With the assumption of time-independence, elastic scattering (i.e., no conversion from one frequency to another in the range of observation), no internal sources (e.g., thermal emission at long wavelengths), and sufficiently rare media that each particle is in the far field of the radiation scattered from any other particle (i.e., no interparticle shadowing), the radiative transfer equation can be written (Chandrasekhar, 1960) for the diffuse intensity $I(\tau,\mu,\phi)$ as

$$2\mu \frac{dI(\tau,\mu,\phi)}{d\tau} = 2I(\tau,\mu,\phi)$$
$$- \frac{1}{2\pi} \int_{-1}^{1} \int_{0}^{2\pi} p(\mu,\phi; \mu',\phi')I(\tau,\mu',\phi')d\phi'd\mu'$$
$$- \frac{1}{2}Fp(\mu, \phi; -\mu_0, \phi_0)e^{-\tau/\mu_0}. \quad (1)$$

The spherical coordinates $\theta = \arccos\mu$ and ϕ, with θ measured from the positive (outward) surface normal, refer to the direction of a pencil of light rays of intensity I at optical depth τ, πF is the incident collimated flux crossing unit area perpendicular to the propagation direction defined by $\theta_0 = \arccos -\mu_0$ and ϕ_0, and $p(\mu,\phi; \mu',\phi')$ is the phase function or single-particle scattering law for radiation scattered from the direction (μ',ϕ') into the direction (μ,ϕ). The phase function is normalized according to the expression

$$\frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} p(\mu,\phi; \mu',\phi')d\phi d\mu = \omega_0, \quad (2)$$

where ω_0 is the single-scattering albedo, i.e., the ratio of the scattering coefficient to the sum of the scattering and absorption coefficients.

If $p(\mu,\phi; \mu',\phi')$ is a function only of the cosine of the scattering angle, Chandrasekhar (1960) has shown that the azimuthal integral satisfies

$$p(\mu,\mu') = \frac{1}{2\pi} \int_{0}^{2\pi} p(\mu,\phi; \mu',\phi')d\phi$$
$$= \omega_0 \sum_{l} (2l + 1)g_l P_l(\mu)P_l(\mu'), \quad (3)$$

where $P_l(\mu)$ is the lth order Legendre polynomial and the coefficients g_l are defined by

$$g_l = \frac{1}{2\omega_0} \int_{-1}^{1} P_l(\mu)p(\mu,1)d\mu. \quad (4)$$

The normalization condition (2) then becomes

$$\int_{-1}^{1} p(\mu,\mu')d\mu = 2\omega_0. \quad (5)$$

With the additional definition

$$I(\tau,\mu) = \int_{0}^{2\pi} I(\tau,\mu,\phi)d\phi, \quad (6)$$

the azimuthal integration of Eq. (1) yields

$$2\mu \frac{dI(\tau,\mu)}{d\tau}$$
$$= 2I(\tau,\mu) - \int_{-1}^{1} p(\mu,\mu')I(\tau,\mu')d\mu'$$
$$- \pi Fp(\mu, -\mu_0)e^{-\tau/\mu_0}. \quad (7)$$

As explained in the Introduction, it is convenient to define the hemispheric integrals

$$I^{\pm}(\tau) = \int_{0}^{1} \mu I(\tau, \pm\mu)d\mu \quad (8)$$

and the quantity

$$\beta_i = \frac{1}{2\omega_0} \int_{0}^{1} p(\mu_i, -\mu')d\mu'$$
$$= 1 - \frac{1}{2\omega_0} \int_{0}^{1} p(\mu_i,\mu')d\mu' \quad (9)$$

in order to produce the following pair of equations from the μ integration of Eq. (7) as it stands and Eq. (7) with μ replaced by $-\mu$:

$$\frac{dI^+}{d\tau} = \int_{0}^{1} I(\tau,\mu)d\mu$$
$$- \frac{1}{2} \int_{0}^{1} \int_{-1}^{1} p(\mu,\mu')I(\tau,\mu')d\mu'd\mu$$
$$- \pi F\omega_0\beta_0 e^{-\tau/\mu_0} \quad (10)$$

$$\frac{dI^-}{d\tau} = - \int_0^1 I(\tau, -\mu)d\mu$$

$$+ \frac{1}{2} \int_0^1 \int_{-1}^1 p(\mu, -\mu')I(\tau,\mu')d\mu'd\mu$$

$$+ \pi F\omega_0(1 - \beta_0)e^{-\tau/\mu_0}. \quad (11)$$

Two-stream methods are defined for present purposes as methods satisfying the simplified expressions

$$\frac{dI^+}{d\tau} = \gamma_1 I^+ - \gamma_2 I^- - \pi F\omega_0\gamma_3 e^{-\tau/\mu_0}, \quad (12)$$

$$\frac{dI^-}{d\tau} = \gamma_2 I^+ - \gamma_1 I^- + \pi F\omega_0\gamma_4 e^{-\tau/\mu_0}, \quad (13)$$

which are obtained from Eqs. (10) and (11) by assuming the μ dependence of I and approximating the integrals. The γ_i's are determined by the approximations used and are independent of τ in all cases. As will be shown, their values are constrained by physical requirements; for example, the constraint $\gamma_3 + \gamma_4 = 1$ follows immediately from energy conservation.

3. Solutions for plane-parallel atmospheres

Eqs. (12) and (13) can be solved by standard techniques for given boundary conditions. In particular, the solutions for collimated radiation incident on a plane-parallel atmosphere with boundary conditions $I^+(\tau') = I^-(0) = 0$, where τ' is the optical thickness, yield the following results for the plane albedo (or reflectance) $R = (\pi F\mu_0)^{-1}I^+(0)$ and the transmittance $T = \exp(-\tau'/\mu_0) + (\pi F\mu_0)^{-1}I^-(\tau')$:

$$R = \frac{\omega_0}{(1 - k^2\mu_0^2)[(k + \gamma_1)e^{k\tau'} + (k - \gamma_1)e^{-k\tau'}]}$$

$$\times [(1 - k\mu_0)(\alpha_2 + k\gamma_3)e^{k\tau'}$$

$$- (1 + k\mu_0)(\alpha_2 - k\gamma_3)e^{-k\tau'}$$

$$- 2k(\gamma_3 - \alpha_2\mu_0)e^{-\tau'/\mu_0}] \quad (14)$$

$$T = e^{-\tau'/\mu_0}$$

$$\times \left\{ 1 - \frac{\omega_0}{(1 - k^2\mu_0^2)[(k + \gamma_1)e^{k\tau'} + (k - \gamma_1)e^{-k\tau'}]} \right.$$

$$\times [(1 + k\mu_0)(\alpha_1 + k\gamma_4)e^{k\tau'}$$

$$- (1 - k\mu_0)(\alpha_1 - k\gamma_4)e^{-k\tau'}$$

$$\left. -2k(\gamma_4 + \alpha_1\mu_0)e^{\tau'/\mu_0}] \right\}. \quad (15)$$

Additional parameters are

$$\alpha_1 = \gamma_1\gamma_4 + \gamma_2\gamma_3, \quad (16)$$

$$\alpha_2 = \gamma_1\gamma_3 + \gamma_2\gamma_4, \quad (17)$$

$$k = (\gamma_1^2 - \gamma_2^2)^{1/2}. \quad (18)$$

For thin atmospheres, i.e., R and T linear in τ', Eqs. (14) and (15) reduce to

$$R = \frac{\omega_0\tau'}{\mu_0} \gamma_3 = 1 - T - \frac{\tau'}{\mu_0} [1 - \omega_0(\gamma_3 + \gamma_4)], \quad (19)$$

which should be compared with the corresponding result

$$R = \frac{\omega_0\tau'}{\mu_0} \beta_0 = 1 - T - \frac{\tau'}{\mu_0} (1 - \omega_0) \quad (20)$$

obtained from Eq. (7). Eqs. (14) and (15) thus approach the correct thin-atmosphere limit if $\gamma_3 = \beta_0$ and if

$$\gamma_3 + \gamma_4 = 1, \quad (21)$$

the latter being a statement of conservation of energy that will be required of all two-stream approximations. Although both of these conditions also follow from the comparison of Eqs. (10) and (11) with Eqs. (12) and (13), some of the two-stream methods involve approximations to the integral β_0 defined by Eq. (9); hence, the condition $\gamma_3 = \beta_0$ is not met by all methods and the symbols γ_3 and γ_4 will be retained for generality.

A useful result applicable to conservative atmospheres follows from the semi-infinite limit

$$R(\tau' = \infty) = \frac{\omega_0(\alpha_2 + k\gamma_3)}{(1 - k\mu_0)(k + \gamma_1)} \quad (22)$$

of Eq. (14). The use of Eqs. (17), (18) and (21) in this expression, together with the requirement that $R(\tau' = \infty) = 1$ when $\omega_0 = 1$, yields

$$\gamma_1 = \gamma_2, \quad \omega_0 = 1. \quad (23)$$

In addition to providing a second condition to be required of all two-stream approximations, Eq. (23) permits the following reduction of Eqs. (14) and (15) for conservative scattering:

$$R(\omega_0 = 1)$$

$$= \frac{1}{1 + \gamma_1\tau'} [\gamma_1\tau' + (\gamma_3 - \gamma_1\mu_0)(1 - e^{-\tau'/\mu_0})]$$

$$= 1 - T(\omega_0 = 1). \quad (24)$$

A second set of two-stream approximations, the results of which are denoted by R' and T', corresponds to Eqs. (12) and (13) without the terms explicitly containing the incident flux πF. The intensities then refer to the total radiation field instead of just the diffuse component. Although the boundary condition at the lower surface of a plane-parallel atmosphere remains $I^+(\tau') = 0$ for the solution of the coupled equations, the boundary condition at the upper surface becomes $I^-(0) = \pi F\mu_0$. The failure of this procedure to require the more detailed condition that $I(\tau, -\mu)$ be a delta function centered on

TABLE 1. Coefficients γ_i in the two-stream equations (12) and (13).

Method	γ_1	γ_2	γ_3
Eddington	$\frac{1}{4}[7 - \omega_0(4 + 3g)]$	$-\frac{1}{4}[1 - \omega_0(4 - 3g)]$	$\frac{1}{4}(2 - 3g\mu_0)$
Modified Eddington	$\frac{1}{4}[7 - \omega_0(4 + 3g)]$	$-\frac{1}{4}[1 - \omega_0(4 - 3g)]$	β_0
Quadrature	$(3^{1/2}/2)[2 - \omega_0(1 + g)]$	$(3^{1/2}\omega_0/2)(1 - g)$	$\frac{1}{2}(1 - 3^{1/2}g\mu_0)$
Modified quadrature*	$3^{1/2}[1 - \omega_0(1 - \beta_1)]$	$3^{1/2}\omega_0\beta_1$	β_0
Hemispheric constant	$2[1 - \omega_0(1 - \beta)]$	$2\omega_0\beta$	β_0
Delta function	$\mu_0^{-1}[1 - \omega_0(1 - \beta_0)]$	$\omega_0\beta_0/\mu_0$	β_0
Hybrid modified Eddington-delta function	$\dfrac{7 - 3g^2 - \omega_0(4 + 3g) + \omega_0g^2(4\beta_0 + 3g)}{4[1 - g^2(1 - \mu_0)]}$	$-\dfrac{1 - g^2 - \omega_0(4 - 3g) - \omega_0g^2(4\beta_0 + 3g - 4)}{4[1 - g^2(1 - \mu_0)]}$	β_0

* β_1 corresponds to $\mu_1 = 3^{-1/2}$.

μ_0 when $\tau = 0$ may be a cause of concern for some of the methods to be discussed in the next section. The plane albedos and transmittances for this second set of two-stream approximations are given by the following expressions analogous to Eqs. (14), (15), (19), (22) and (24):

$$R' = \frac{I^+(0)}{I^-(0)} = \frac{\gamma_2[1 - \exp(-2k\tau')]}{k + \gamma_1 + (k - \gamma_1)\exp(-2k\tau')}, \quad (25)$$

$$T' = \frac{I^-(\tau')}{I^-(0)} = \frac{2k\exp(-k\tau')}{k + \gamma_1 + (k - \gamma_1)\exp(-2k\tau')}, \quad (26)$$

$$R'(\text{thin atmos}) = \gamma_2\tau'$$
$$= 1 - T'(\text{thin atmos})$$
$$- (\gamma_1 - \gamma_2)\tau', \quad (27)$$

$$R'(\tau' = \infty) = \frac{\gamma_2}{k + \gamma_1}, \quad (28)$$

and, if $\gamma_1 = \gamma_2$ when $\omega_0 = 1$,

$$R'(\omega_0 = 1) = \frac{\gamma_1\tau'}{1 + \gamma_1\tau'} = 1 - T'(\omega_0 = 1). \quad (29)$$

Eq. (27) yields the correct thin-atmosphere limit (20) for collimated incidence only if $\gamma_2 = \omega_0\beta_0/\mu_0$ and $\gamma_1 - \gamma_2 = (1 - \omega_0)/\mu_0$, which conditions are seldom satisfied in the models to be considered.

4. Two-stream approximations

The coefficients γ_i are determined for each two-stream approximation according to the following steps: 1) the assumption of an approximate form for $I(\tau,\mu)$ that defines the particular model, 2) the use of this form to evaluate the integrals in Eqs. (10) and (11), and 3) the direct comparison of the results with Eqs. (12) and (13). Expressions for γ_1, γ_2 and γ_3 are collected in Table 1 and γ_4 is understood to be $1 - \gamma_3$ unless noted otherwise in the text.

a. Eddington and modified Eddington approximations

Both the standard Eddington approximation (cf. Irvine, 1968, 1975; Kawata and Irvine, 1970; Shettle

and Weinman, 1970) and the modified Eddington approximation (as introduced in this paper) start with the assumption $I(\tau,\mu) = I_0(\tau) + \mu I_1(\tau)$, the use of which in Eq. (8) gives

$$I(\tau, \pm\mu) = \frac{1}{2}[(2 \pm 3\mu)I^+(\tau) + (2 \mp 3\mu)I^-(\tau)],$$
$$\mu \geqslant 0. \quad (30)$$

The substitution of this expression in the integrands in Eqs. (10) and (11) yields

$$\int_0^1 I(\tau, \pm\mu)d\mu = \frac{1}{4}[(4 \pm 3)I^+ + (4 \mp 3)I^-], \quad (31)$$

$$\frac{1}{2}\int_0^1 \int_{-1}^1 p(\mu, \pm\mu')I(\tau,\mu')d\mu'd\mu$$
$$= \frac{1}{4}\omega_0[(4 \pm 3g)I^+ + (4 \mp 3g)I^-], \quad (32)$$

and finally the γ_1 and γ_2 shown in the first two rows of Table 1 after comparisons with Eqs. (12) and (13). Employed in the evaluation of the integral in Eq. (32) are Eq. (3), the orthogonality of Legendre polynomials, the normalization condition (5) requiring $g_0 = 1$, and the definition of the asymmetry factor $g = g_1$. Note that the higher order anisotropies of p do not appear in this approximation of the multiple-scattering integral.

Also shown in Table 1 are the values of γ_3 for the standard and modified Eddington approximations, the former corresponding to the substitution of $p(\mu_0, -\mu') = \omega_0(1 - 3g\mu_0\mu')$ for the actual phase function in Eq. (9). One apparent advantage of the standard method is the consistency provided between the forms of γ_3 and Eq. (32); however, such consistency may not be essential if higher order contributions to p are effectively smoothed by multiple scattering. An argument would then exist for describing the multiple-scattering integral by Eq. (32) and simultaneously using the full phase function to evaluate the single-scattering term involving γ_3, as is done in the modified approximation. In any event, as seen from Eq. (19) and Table 1, a definite disadvantage to the standard γ_3 is the negative plane albedo computed for thin atmospheres when $g > 2(3\mu_0)^{-1}$. Irvine (1968, 1975) has demonstrated
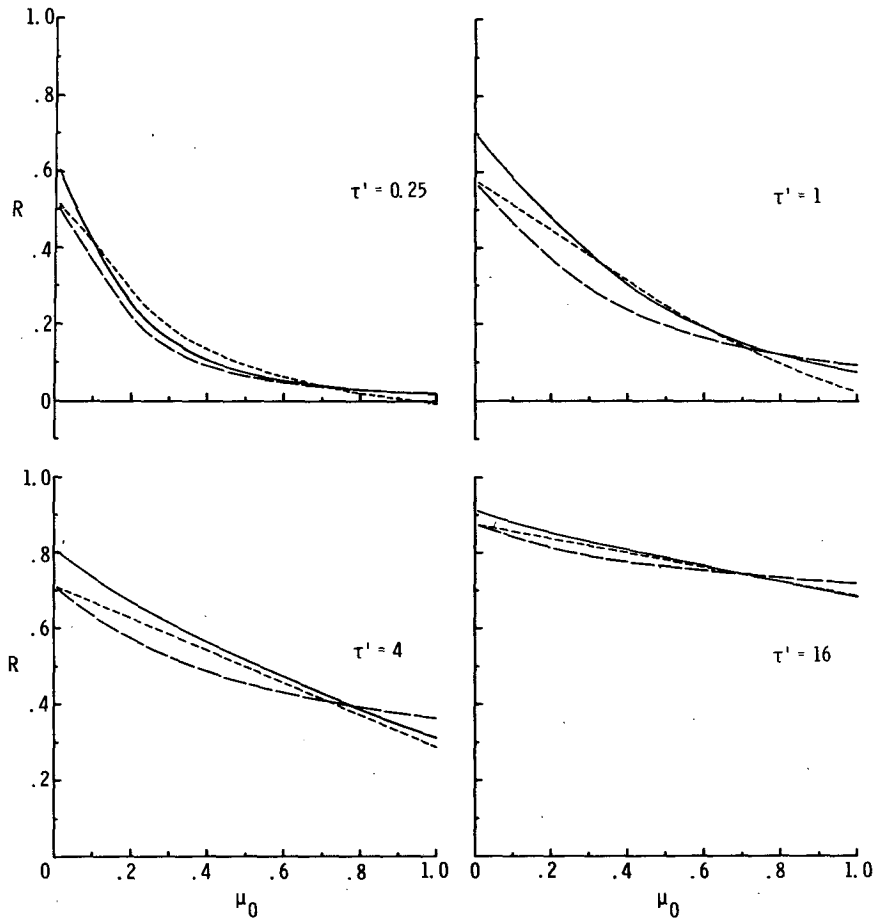
FIG. 1. Plane albedo R as a function of μ_0 (cosine of the angle of incidence) and the optical thickness τ' of a plane-parallel atmosphere for the Eddington (short dashes) and modified Eddington (long dashes) approximations. The scattering is conservative ($\omega_0 = 1$), the phase function is Henyey-Greenstein with $g = 0.75$, and the solid curves are rigorous discrete-ordinate results.

numerically the inadequacies of the standard method when single scattering predominates (e.g., small τ', small ω_0, near-grazing incidence or large g) and thus when γ_3 is especially significant.

The same values of γ_1 and γ_2 are obtained if the intensities refer to the total radiation field. As seen from Table 1 and Eqs. (25), (27) and (28), negative plane albedos result for atmospheres of all optical thicknesses when $g > (4\omega_0 - 1)(3\omega_0)^{-1}$ and thus for all non-negative values of g if $\omega_0 < \frac{1}{4}$. Lyzenga (1973) reached the same conclusion in calculations on semi-infinite atmospheres. Of more importance, however, is the fact that γ_1 and γ_2 are independent of μ_0; hence, the Eddington method described in this paragraph cannot apply to collimated incidence and will not be further considered.

b. Quadrature and modified quadrature methods

Since a number of different two-point quadrature methods have been developed for applications to

radiative transfer, a more detailed discussion is given than might otherwise be justified. All such methods start with the approximation

$$\int_{-1}^{1} p(\mu,\mu')I(\tau,\mu')d\mu' = p(\mu,\mu_1)I(\tau,\mu_1)$$

$$+ p(\mu, -\mu_1)I(\tau, -\mu_1), \quad (33)$$

so that

$$\frac{1}{2} \int_{0}^{1} \int_{-1}^{1} p(\mu, \pm\mu')I(\tau,\mu')d\mu'd\mu$$

$$= \frac{\omega_0}{\mu_1}\left[\left(\frac{1 - \beta_1}{\beta_1}\right)I^+ + \left(\frac{\beta_1}{1 - \beta_1}\right)I^-\right] \quad (34)$$

if $I^\pm(\tau) = \mu_1 I(\tau, \pm\mu_1)$. The quadrature points are $\pm\mu_1$, β_1 is defined by Eq. (9), and the upper and lower entries in the square brackets in Eq. (34) correlate with the plus and minus signs in the integrand. These expressions can also be identified with the fol-

lowing assumption about $I(\tau,\mu)$ that will be useful for later developments:

$$I(\tau, \pm\mu) = I(\tau, \pm\mu_1)\delta(\mu - \mu_1)$$

$$= \mu_1^{-1}I^{\pm}(\tau)\delta(\mu - \mu_1), \quad \mu \geq 0. \quad (35)$$

The substitution into Eqs. (10) and (11) of Eqs. (33) and (34) and the approximation

$$\int_0^1 I(\tau, \pm\mu)d\mu = I^{\pm}/\mu_1 \quad (36)$$

yields

$$\gamma_1 = \frac{1}{\mu_1}[1 - \omega_0(1 - \beta_1)], \quad (37)$$

$$\gamma_2 = \omega_0\beta_1/\mu_1, \quad (38)$$

$$\gamma_3 = \beta_0, \quad (39)$$

$$\gamma_4 = 1 - \beta_0, \quad (40)$$

after comparisons of the results with Eqs. (12) and (13). With particular choices of μ_1, Eqs. (37)–(40) can be made to represent any of the two-stream methods based on two-point quadrature.

Liou's (1973, 1974) method of discrete ordinates employs the Gaussian choice $\mu_1 = 3^{-1/2}$, which makes Eq. (33) an accurate representation of the multiple-scattering integral if the integrand is adequately described by a cubic in μ'. However, in order to duplicate Liou's complete formulation, quadrature must also be applied to the integrals β_0 and β_1 defined by Eq. (9) and the phase function must be replaced everywhere by the summation (3) with $l = 0$ and 1. For example,

$$\gamma_3 = \frac{1}{2\omega_0}p(\mu_1, -\mu_0) = \frac{1}{2}(1 - 3^{1/2}g\mu_0), \quad (41)$$

$$\gamma_4 = \frac{1}{2\omega_0}p(\mu_1,\mu_0) = \frac{1}{2}(1 + 3^{1/2}g\mu_0). \quad (42)$$

The truncation of the phase-function summation is necessary for the condition (21) of conservation of energy to be satisfied. Liou's method is denoted as quadrature in Table 1, where the complete set of γ_i coefficients is given.

Eqs. (19) and (41) show that Liou's two-point method yields negative plane albedos for thin atmos-
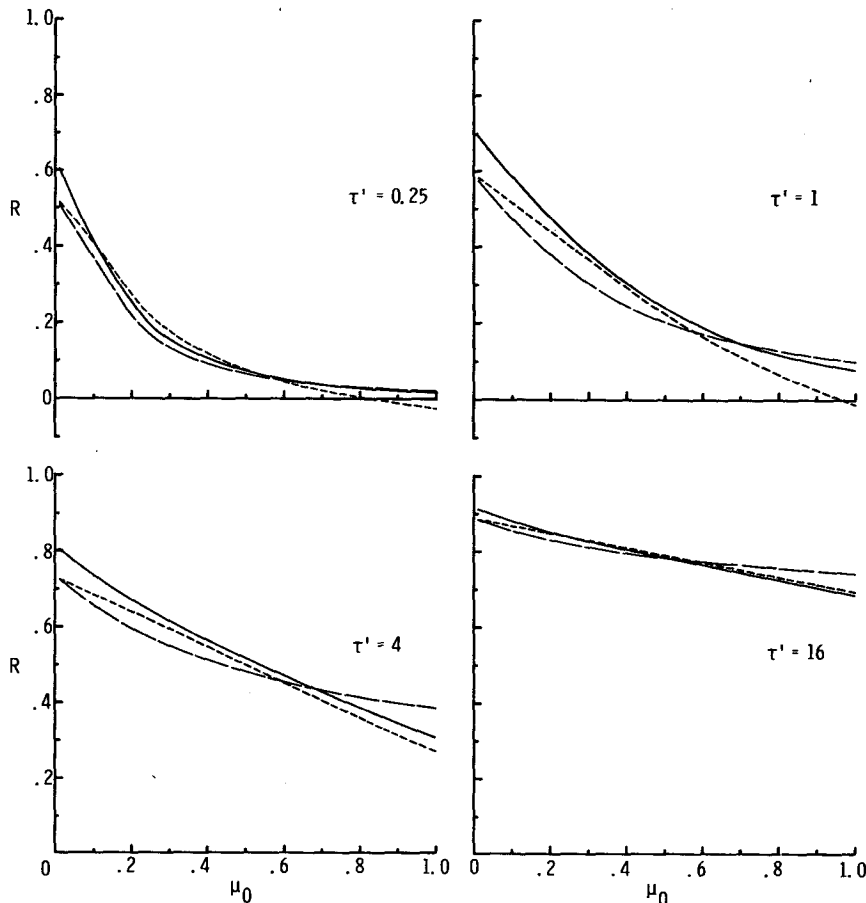

FIG. 2. As in Fig. 1 except the short-dash curves refer to two-point quadrature and the long-dash curves to modified two-point quadrature.
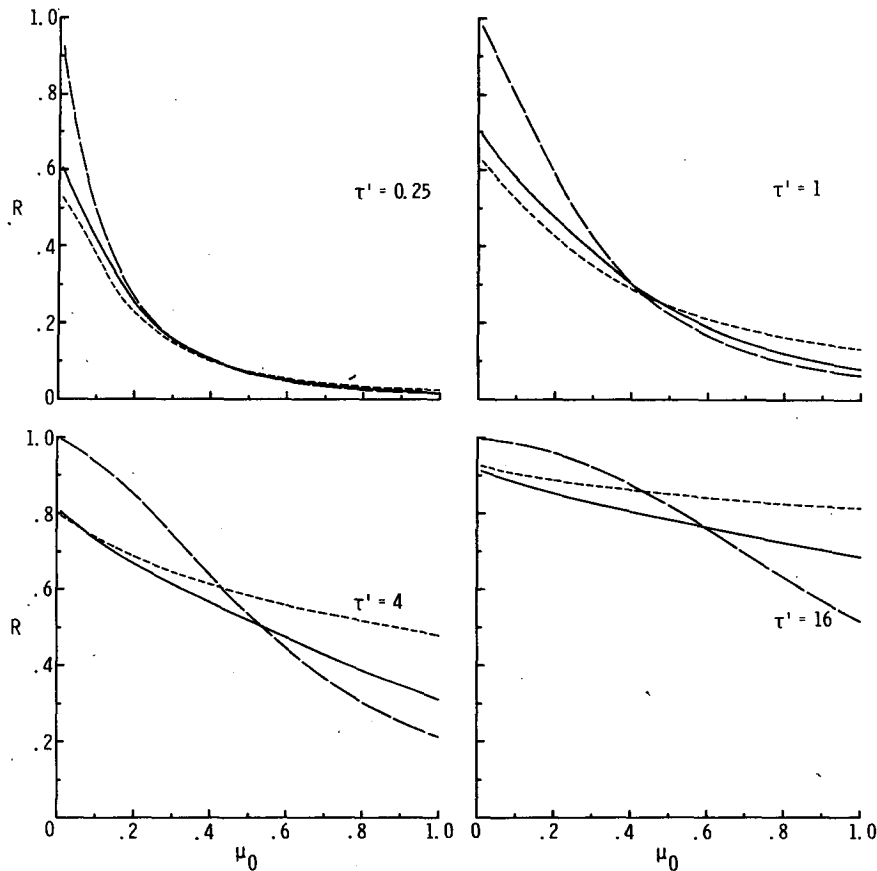
FIG. 3. As in Fig. 1 except the short-dash curves refer to the hemispheric-constant method
and the long-dash curves to the delta-function method.

pheres when $g > (3^{1/2}\mu_0)^{-1}$. Such behavior, which is similar to that of the standard Eddington approximation, can be avoided by using the full phase functions in evaluations of β_0 and β_1. The results are denoted in the fourth row of Table 1 as modified quadrature. It should be mentioned that in the quadrature formulations of this paper and also in the development by Liou, since he ultimately has to evaluate $\int_0^1 \mu I(\tau, \pm\mu)d\mu$ in order to obtain R and T, $\mu_1 = 3^{-1/2}$ is not the appropriate choice for ensuring accuracy to the maximum polynomial degree for the integrands in the integrals with limits 0 and 1. No compelling reason thus exists for selecting $\mu_1 = 3^{-1/2}$ in the *overall* problem; in fact, the choice $\mu_1 = \mu_0$ in Eq. (35) will form the basis of the delta-function method to be discussed in Section 4d.

Four additional quadrature (or quadrature-like) methods, all of which interpret $I(\tau,\mu)$ as the total radiation field, have frequently been employed. Eqs. (25)–(29) provide the plane albedos and transmittances for plane-parallel atmospheres in each case and γ_1 and γ_2 are given by Eqs. (37) and (38), respectively.

One such method was formulated by Sagan and

Pollack (1967) and subsequently clarified by Lyzenga (1973). It assumes $\mu_1 = 3^{-1/2}$ and uses the approximation

$$\beta_1 \approx \frac{1}{2\omega_0} p(3^{-1/2}, -3^{-1/2}) \approx \frac{1}{2}(1 - g); \quad (43)$$

consequently, the plane albedo and transmittance do not depend on μ_0 and the method is thus inappropriate for collimated incidence.

The second and third of these additional methods are called two-stream and modified two-stream approximations by Irvine (1968, 1975), with derivational credit given to Chu and Churchill (1955) for the former and to Sagan and Pollack (1967) for the latter. Each approximation introduces a μ_0 dependence not present in the work of Sagan and Pollack or Lyzenga by employing μ_0 for μ_1 where it explicitly appears in Eqs. (37) and (38). The approximations differ from each other by the use in Eqs. (37) and (38) of the expression

$$\beta_1 = \frac{1}{2\omega_0} \int_0^1 p(\mu, -1)d\mu \quad (44)$$

in the two-stream method and the Sagan-Pollack approximation (43) in the modified two-stream method.

Difficulties common to both the two-stream and the modified two-stream approximations are illustrated by the combination of Eqs. (27) and (38) to give thin-atmosphere limits that differ from the exact answer (20) by the substitution of β_1 for β_0. This discrepancy can be resolved by setting μ_1 equal to μ_0 everywhere in the quadrature development so that $\beta_1 = \beta_0$ automatically. The results summarized by Eqs. (25)–(29), (37) and (38) are then exactly equivalent to Coakley and Chýlek's model 1 (1975), although their derivation is entirely different. Further discussion is given in Section 4d.

c. Hemispheric constant method

Coakley and Chýlek (1975) in their model 2 assumed constant values for $I(\tau,\mu)$ and $I(\tau, -\mu)$, the translation of which to the present notation yields $I(\tau, \pm\mu) = 2I^{\pm}(\tau)$ according to Eq. (8). The integrals in Eqs. (10) and (11) can thus be written

$$\int_0^1 I(\tau, \pm\mu)d\mu = 2I^{\pm} \qquad (45)$$

$$\frac{1}{2} \int_0^1 \int_{-1}^1 p(\mu, \pm\mu')I(\tau,\mu')d\mu'd\mu$$

$$= 2\omega_0\left[\left(\frac{1-\beta}{\beta}\right)I^+ + \left(\frac{\beta}{1-\beta}\right)I^-\right], \qquad (46)$$

where

$$\beta = \frac{1}{2\omega_0} \int_0^1 \int_0^1 p(\mu, -\mu')d\mu d\mu'. \qquad (47)$$

Comparisons of these results with Eqs. (12) and (13) give the expressions for γ_1, γ_2 and γ_3 listed in the fifth row of Table 1.

It should be noted that the motivation behind each of Coakley and Chýlek's two models was to produce equations having the correct thin-atmosphere limits. As seen from Eqs. (10)–(13), (19) and (20), these limits automatically result from the present format because of the identification of γ_3 with β_0 and γ_4 with $1 - \beta_0$ for any of the two-stream methods that interpret $I(\tau,\mu)$ as the diffuse component of the radiation field. Difficulties with the limits appeared in the standard Eddington and quadrature (Liou) approximations mainly because truncated phase functions were used in the evaluations of β_0.
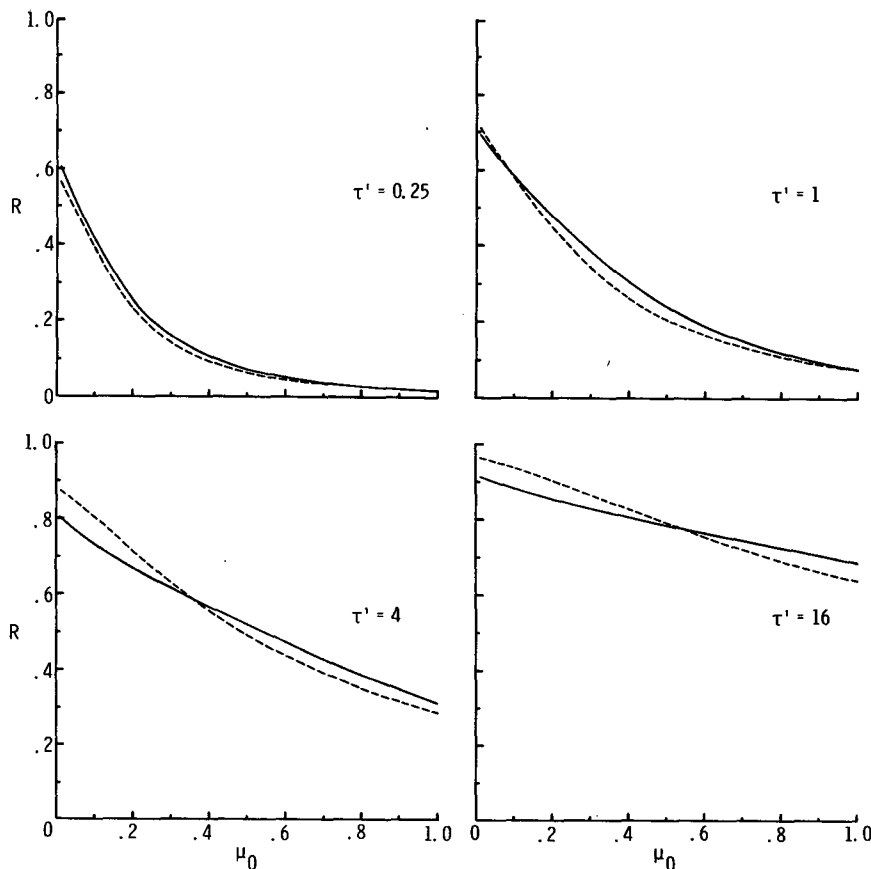

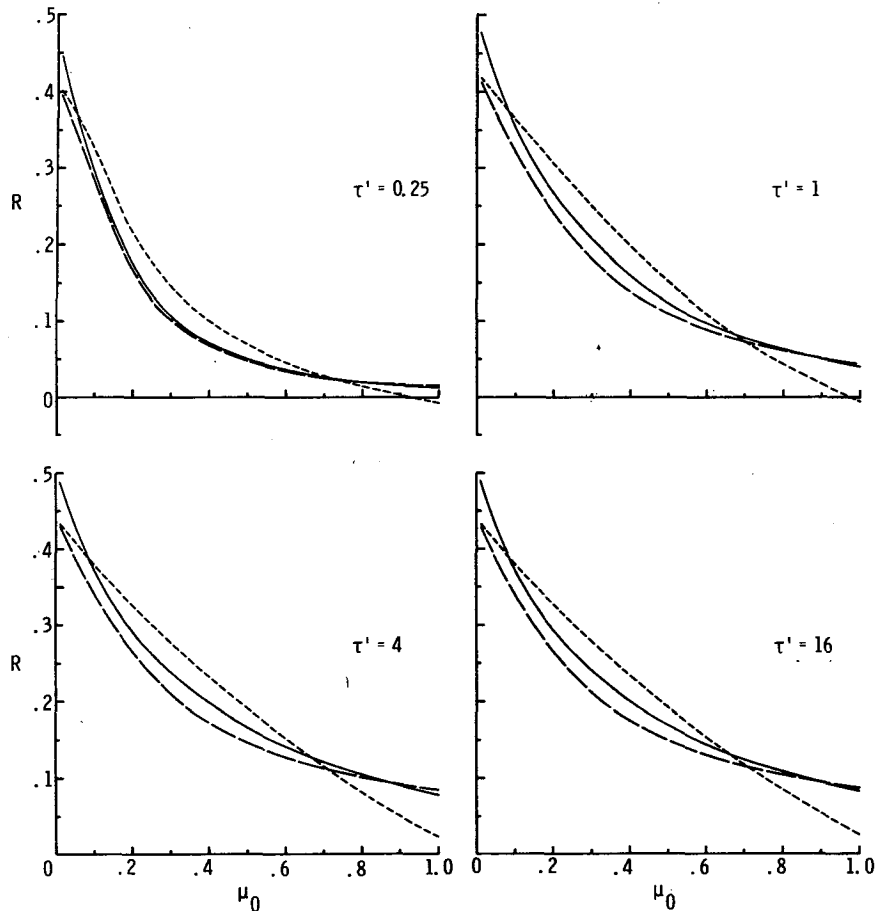FIG. 4. As in Fig. 1 except the dashed curves refer to the hybrid approximation.

FIG. 5. As in Fig. 1 except $\omega_0 = 0.8$. Eddington (short dash), modified Eddington (long dash).

d. Delta-function method

The delta-function method defined in this paper refers specifically to the substitution of μ_0 for μ_1 in Eq. (35). Accordingly, the expressions for γ_1, γ_2 and γ_3 are those of Eqs. (37), (38) and (39), with $\mu_1 = \mu_0$ and $\beta_1 = \beta_0$, and are listed in the sixth row of Table 1. The method is identical in result to model 1 of Coakley and Chýlek (1975) discussed in Section 4b, but differs substantially in the concept and procedure of its derivation.

Eq. (35) with $\mu_1 = \mu_0$ has the important and useful property of being an exact solution (Adamson, 1975) of the radiative transfer equation if the phase function satisfies

$$p(\mu, \pm\mu_0) = \omega_0[(1 - g)\delta(\mu \pm \mu_0)$$
$$+ (1 + g)\delta(\mu \mp \mu_0)]. \quad (48)$$

Since aerosols and water droplets generally scatter very asymmetrically and show some tendencies toward Eq. (48) for large values of the magnitude of g, it may be advantageous for such applications to use hybrid approximations employing the delta-function method as a constituent part. In fact, a

hybridization of the delta-function method with the modified Eddington approximation is the subject of Section 4g and will be shown to give the best overall agreement with exact calculations for $g = 0.75$ in the Henyey-Greenstein phase function (see Section 5). This hybridization differs substantially in concept from those previously proposed.

e. Six-beam model (normal incidence)

Chu and Churchill (1955) developed a general six-beam model that was shown both by themselves and by Emslie and Aronson (1973) to degenerate to a two-beam form in the case of normal incidence on a plane-parallel scattering medium. For simplicity and in order to correspond directly to the work of these authors, we let $I(\tau,\mu,\phi)$ represent the total radiation field in this subsection. The translation of their expressions to the forms of Eqs. (12) and (13) without the πF terms is straightforwardly accomplished and gives

$$\gamma_1 = 1 - \omega_0 S_f - \frac{4\omega_0^2 S_t^2}{1 - \omega_0(S_f + S_b + 2S_t)}, \quad (49)$$

$$\gamma_2 = \omega_0 S_b + \frac{4\omega_0^2 S_t^2}{1 - \omega_0(S_f + S_b + 2S_t)}, \qquad (50)$$

for use in the reflectance and transmittance results (25) and (26). The symbols S_f, S_b and S_t are ratios of forward, backward and transverse scattering cross sections, respectively, to the total scattering cross section of a single particle, so that $S_f + S_b + 4S_t = 1$.

Eqs. (49) and (50) yield the difference

$$\gamma_1 - \gamma_2 = (1 - \omega_0)\left[1 + \frac{4\omega_0 S_t}{1 - \omega_0(1 - 2S_t)}\right], \qquad (51)$$

which requires S_t to vanish in order to satisfy the energy-conservation condition $\gamma_1 - \gamma_2 = 1 - \omega_0$ previously deduced from Eqs. (20) and (27) for physically plausible phase functions (as opposed to the delta-function phase functions required to make the six-beam model an exact solution of the radiative transfer equation, but which led to unrealistic photon trapping in directions parallel to the medium surface). Moreover, the reflectance for optically thin media can be written

$$R' = \omega_0 \tau'\left[S_b + \frac{4\omega_0 S_t^2}{1 - \omega_0(S_f + S_b + 2S_t)}\right], \qquad (52)$$

which generally does not agree well with the exact limit

$$R = \frac{\tau'}{2} \int_0^1 p(\mu, -1)d\mu. \qquad (53)$$

For example, in the case of isotropic scatterers $(S_f = S_b = S_t = \frac{1}{6})$ and small ω_0, the reflectance of Eq. (53) is nearly triple that of Eq. (52).

Existing six-beam models thus do not appear to be as useful for rapid approximations as some of the two-stream methods already considered. In particular, Chu and Churchill's two-beam model with $S_t = 0$, $S_f + S_b = 1$, and

$$S_b = \frac{1}{2\omega_0} \int_0^1 p(\mu, -1)d\mu, \qquad (54)$$

is definitely superior in that it applies for all phase functions and approaches the proper limit for thin atmospheres.
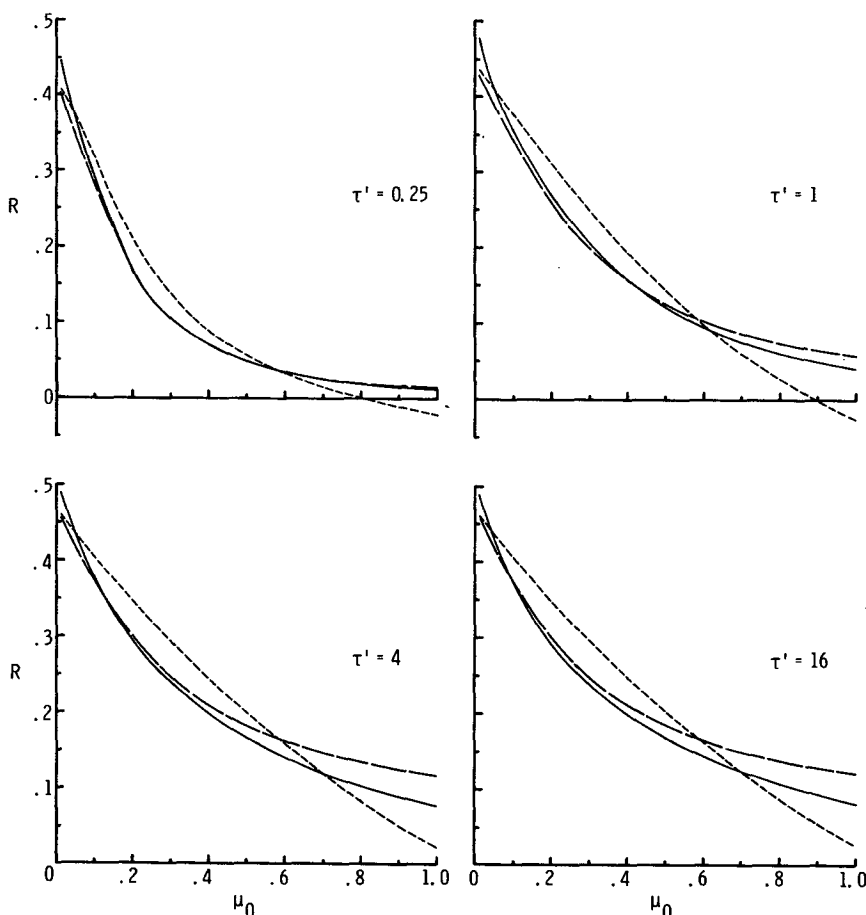

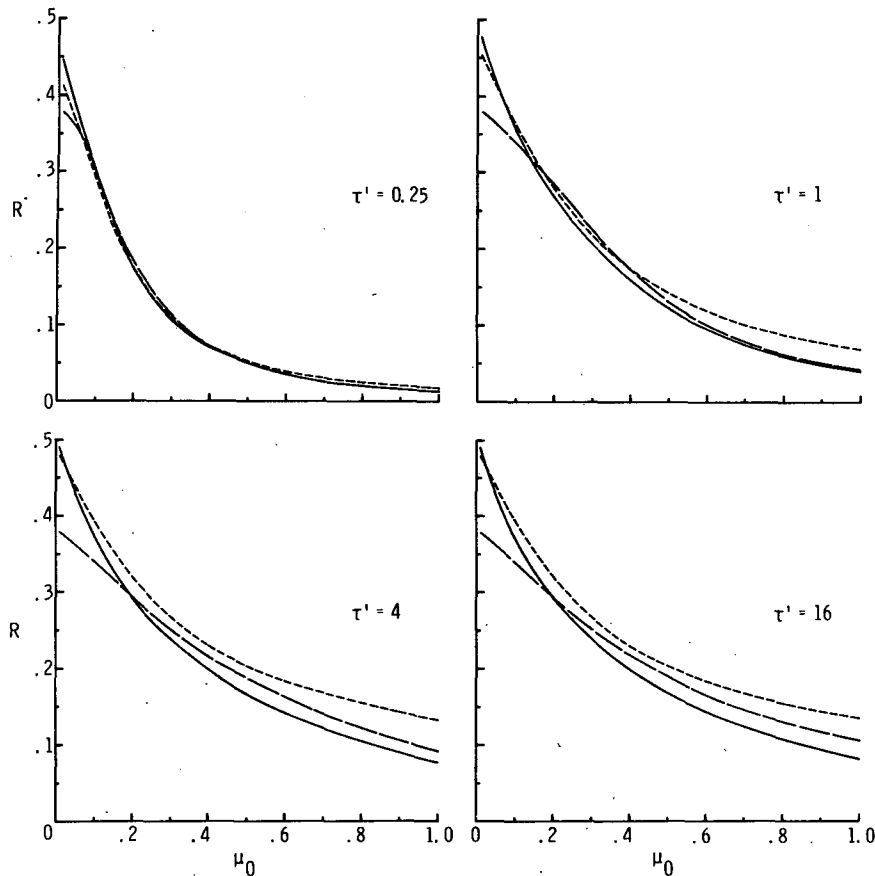FIG. 6. As in Fig. 2 except $\omega_0 = 0.8$. Quadrature (short dash), modified quadrature (long dash).

FIG. 7. As in Fig. 3 except $\omega_0 = 0.8$. Hemispheric constant (short dash), delta function (long dash).

f. Similarity relations

Similarity relations are discussed by Irvine (1975) for converting radiative-transfer results for isotropic phase functions to the corresponding results for anisotropic phase functions. They involve the replacement of ω_0 and τ' in the isotropic equations with $\omega_0(1 - g)(1 - \omega_0 g)^{-1}$ and $\tau'(1 - \omega_0 g)$, respectively, so that the plane albedo for thin atmospheres can be written as follows for any of the methods in Table 1:

$$R(\text{thin atmos}) = \frac{\omega_0 \tau'}{2\mu_0}(1 - g). \qquad (55)$$

While this expression is not accurate for the linear phase function $p(\mu,\mu') = \omega_0(1 + 3g\mu\mu')$ with $|g| \leqslant \frac{1}{3}$, as seen from the exact limit

$$R(\text{thin atmos}) = \frac{\omega_0 \tau'}{2\mu_0}\left(1 - \frac{3g\mu_0}{2}\right), \qquad (56)$$

it is preferable to the standard Eddington and two-point quadrature (Liou) albedos that become negative as g approaches its absolute maximum value of unity for more complex phase functions. An in-

formative comparison can also be made for the Henyey-Greenstein phase function with $\mu_0 = 1$, for which Eq. (20) yields the exact answer

$$R(\text{thin atmos}) = \frac{\omega_0 \tau'(1 - g)}{2g}\left[\frac{1 + g}{(1 + g^2)^{1/2}} - 1\right]$$

$$= \frac{\omega_0 \tau'}{2}(1 - g)\left(1 - \frac{g}{2} + \cdots\right). \qquad (57)$$

Since better results are obtained by most of the methods in Table 1 in the limit of thin atmospheres, similarity relations will not be considered further.

g. Modified Eddington-delta function hybrid method

As mentioned in Section 4d, a combination of the delta-function method that is accurate in the extreme case of highly anisotropic phase functions with a second method that is more accurate for isotropic phase functions may provide significant improvements in two-stream descriptions of aerosols and water droplets. After a number of trial calculations, we have concluded that the best second method for this purpose is the modified Eddington approximation. Accordingly, a reasonable linear combina-

tion of Eqs. (30) and (35) is the following expression that yields the Eddington approximation when $g = 0$, the delta-function method when $g = \pm 1$, and satisfies Eq. (8) for the integrated intensities:

$$I(\tau, \pm\mu) = \frac{1}{1 - g^2(1 - \mu_0)}\left\{(1 - g^2)\left[\left(1 \pm \frac{3\mu}{2}\right)I^+\right.\right.$$
$$\left.\left. + \left(1 \mp \frac{3\mu}{2}\right)I^-\right] + g^2\delta(\mu - \mu_0)I^\pm\right\}. \quad (58)$$

The use of Eq. (58) in evaluating the integrals in Eqs. (10) and (11) and the subsequent comparisons with Eqs. (12) and (13) give the values of γ_1, γ_2 and γ_3 shown in the bottom line of Table 1.

A different delta-Eddington hybrid method has been developed by Joseph et al. (1976). They first assume the phase function

$$p(\mu,\mu') = \omega_0\left[2g^2\delta(\mu - \mu')\right.$$
$$\left. + (1 - g^2)\left(1 + \frac{3g\,\mu\mu'}{1 + g}\right)\right], \quad (59)$$

which peaks in the forward direction and yields values of the first three moments (with respect to

Legendre polynomials) of p that agree with those of the Henyey-Greenstein function. Next they scale the optical depth with the factor $1 - \omega_0 g^2$ in order to remove delta-function terms from the radiative transfer equation and finally they solve the resulting Eddington-like equation by Eddington's method. This approach differs primarily in two respects from the hybrid model proposed here: 1) approximations to both $p(\mu,\mu')$ and $I(\tau,\mu)$ are made in Joseph's method, whereas only $I(\tau,\mu)$ is approximated in our model; and 2) Joseph's expressions cannot give correct thin-atmosphere limits for all phase functions because of the approximation to p. We have numerically compared the two methods for a Henyey-Greenstein function with $g = 0.75$ and found the method of this paper to be generally superior over a range of values of ω_0 and optical thickness.

5. Numerical comparisons

The seven methods listed in Table 1 are compared in this section for the asymmetry factor $g = 0.75$ in the Henyey-Greenstein phase function

$$p(\mu,\phi;\,\mu',\phi') = \omega_0(1 - g^2)\{1 + g^2 - 2g[\mu\mu'$$
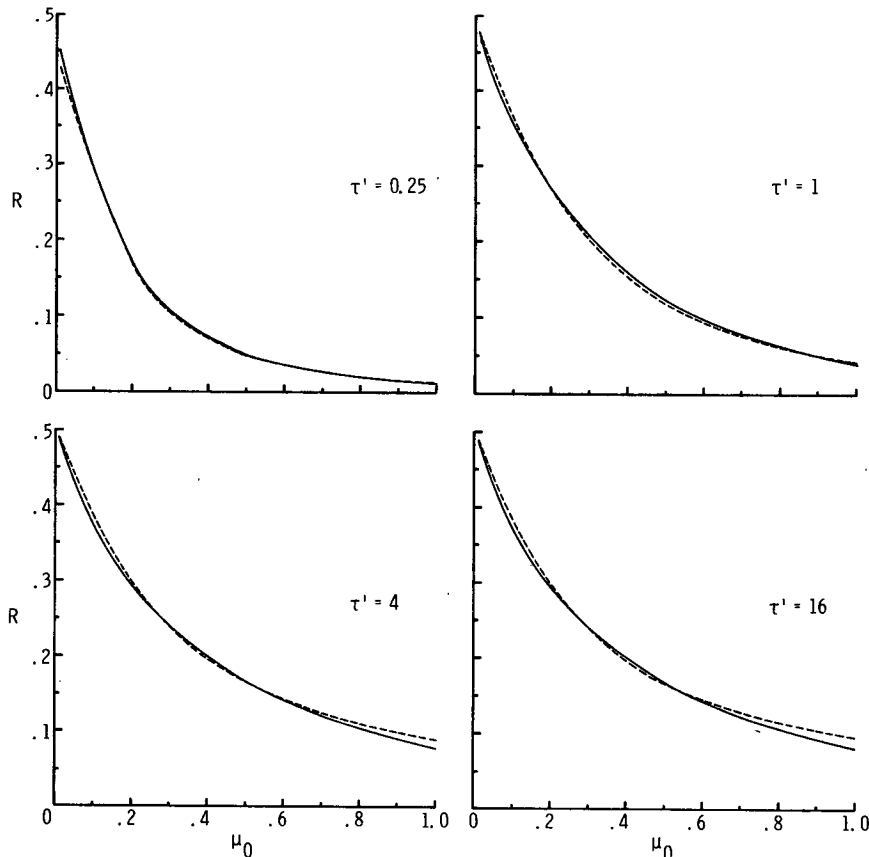$$+ (1 - \mu^2)^{1/2}(1 - \mu'^2)^{1/2}\cos(\phi - \phi')]\}^{-3/2}. \quad (60)$$

FIG. 8. As in Fig. 4 except $\omega_0 = 0.8$. Hybrid method (dashed curves).
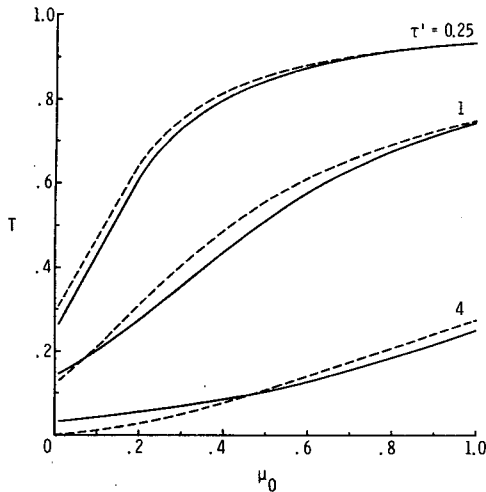
FIG. 9. Transmittance T as a function of μ_0 (cosine of the angle of incidence) and the optical thickness τ' of a plane parallel atmosphere for the hybrid approximation (dashed curves) and rigorous discrete-ordinate calculations (solid curves). The single-scattering albedo ω_0 is 0.8 and the phase function is Henyey-Greenstein with asymmetry factor $g = 0.75$.

Both the phase function and this value of g were used by Liou (1973) as reasonably representative of atmospheric clouds and hazes. It may be noted (see Van de Hulst, 1968) that the ϕ-integration of Eq. (60) yields the Legendre-polynomial expansion (3) of $p(\mu,\mu')$ with the coefficients g_l equal to g^l.

Figs. 1–4 show the plane albedo for conservative scattering ($\omega_0 = 1$) as a function of μ_0 and the optical thickness τ' of a plane-parallel atmosphere. The solid curve in each case represents results that agree, where comparisons are possible, to four significant figures with the results of Liou (1973) and that are obtained by a discrete-ordinate method using 80–100 Gaussian composite quadrature points. Except for the hemispheric-constant and delta-function methods in Fig. 3 and the negative albedos for thin atmospheres (small τ'/μ_0) computed from the standard Eddington and two-point quadrature (Liou) approximations, none of the two-stream curves appear totally inadequate for many applications. An interesting result from Figs. 1 and 2 is the similarity between the standard Eddington and quadrature curves and between the modified Eddington and modified quadrature curves, in spite of the fact that the expressions for γ_1 and γ_2 in Table 1 are quite different. In addition, the superiority shown for thin atmospheres by the modified techniques in both figures has clearly disappeared for large optical thicknesses.

Also inferior to the standard Eddington and quadrature methods for large τ' is the hybrid approximation shown in Fig. 4. However, there is some justification for a claim of overall superiority of the hybrid method for the following reasons: 1) it yields reasonably accurate albedos for all of the optical thick-

nesses considered; 2) it gives the best results for small τ' (not excluding grazing incidence); 3) it should produce excellent Bond albedos from moderately large to large optical thicknesses because the errors for $\tau' = 4$ and 16 in Fig. 4 will tend to cancel in integrations of R over μ_0; 4) it reduces in the limit $g = 0$ to the Eddington approximation, which is assumed to be adequate for isotropic scattering; and 5) it reduces in the limits $g = \pm1$ to the exact solution of the radiative transfer equation. Advantage 5) is especially important for the strongly directed phase functions associated with atmospheric aerosols [e.g., Hansen (1969) and Coakley and Chýlek (1975) used $g = 0.844$ as appropriate for water clouds, the Henyey-Greenstein function being very needle-shaped in this case]. The inadequacies in extreme cases of the standard Eddington and two-point quadrature (Liou) approximations are illustrated by the results ($\omega_0 = g = 1$, τ' finite)

$$R(\text{Eddington}) = \frac{1}{2}\left(1 - \frac{3\mu_0}{2}\right)(1 - e^{-\tau'/\mu_0}), \quad (61)$$

$$R(\text{quadrature}) = \frac{1}{2}(1 - 3^{1/2}\mu_0)(1 - e^{-\tau'/\mu_0}), \quad (62)$$

which differ substantially from the exact albedo $R = 0$.

Figs. 5–8 are the same as Figs. 1–4 except for $\omega_0 = 0.8$ and the doubling of the vertical scale. Several conclusions are different in this example of nonconservative scattering from what they were in the conservative case: (i) the modified Eddington and modified quadrature methods (Figs. 5 and 6) maintain their superiority over the corresponding standard approximations for all values of τ' considered; (ii) the standard Eddington and quadrature results are again similar to each other, but neither are they as accurate as when $\omega_0 = 1$ nor do they approach the correct values for large optical thicknesses; and (iii) the hemispheric-constant and delta-function methods in Fig. 7 are much improved over what they were for $\omega_0 = 1$, even surpassing in some cases the standard Eddington and quadrature methods. Most important is the excellent agreement shown in Fig. 8 between the hybrid and the accurate discrete-ordinate calculations for all values of τ'. This agreement clearly enhances the previous claim of overall superiority for the new hybrid approximation, at least for the examples considered, since the competition provided by the standard Eddington and quadrature methods for conservative scattering does not survive the transition to $\omega_0 = 0.8$.

Fig. 9 shows that transmittances calculated by the hybrid approximation for $g = 0.75$, $\omega_0 = 0.8$ and several values of τ' are in satisfactory agreement with rigorous results. Good agreement is also obtained for the conservative case $\omega_0 = 1$, where conservation of energy requires errors in T to be of the same order as those in R.

As a final note, analytical comparisons can be

made (Table 1) between the expressions for γ_1, γ_2 and γ_3 in the various two-stream approximations. However, except for the curious behavior of the Eddington γ_2 (e.g., negative values for small ω_0 and failure to be directly proportional to ω_0 so that diffuse I^- radiation contributes in this model to changes in I^+ without additional scattering), little of apparent consequences seems to be gained by such comparisons. A more profitable use of Table 1 may be the direct determination of the generalized Kubelka-Munk coefficients (Duntley, 1942; Mudgett and Richards, 1971, 1972; Brinkworth, 1972) that have been employed for many years in numerous industrial applications.

6. Concluding remarks

A system of coupled differential equations has been developed that applies to existing two-stream approximations and thus provides a convenient framework for direct comparisons of methods and results. The various methods were derived and discussed within this framework and several modifications to standard techniques were introduced for the purpose of obtaining the best results in the limit of optically thin atmospheres (including aerosol layers). Numerical comparisons over wide ranges of atmospheric and illumination conditions showed that a new hybrid approximation is superior overall, at least for the examples considered, to the existing methods. Although solutions to the radiative transfer equations were obtained only for plane-parallel atmospheres, extensions to reflecting boundaries are easily found from appropriate manipulations of the results presented (Chandrasekhar, 1960, p. 279). Extensive applications of the hybrid approximation to atmospheres with aerosols, clouds or haze should thus be feasible.

REFERENCES

Adamson, D., 1975: The role of multiple scattering in one-dimensional radiative transfer. NASA Tech. Note D-8084. [NTIS N76-12823].

Brinkworth, B. J., 1972: Interpretation of the Kubelka-Munk coefficients in reflection theory. *Appl. Opt.*, 11, 1434–1435.

Chandrasekhar, S., 1960: *Radiative Transfer.* Dover, 393 pp.

Chu, C. M., and S. W. Churchill, 1955: Numerical solution of problems in multiple scattering of electromagnetic radiation. *J. Phys. Chem.*, 59, 855–863.

Coakley, J. A., Jr., and P. Chýlek, 1975: The two-stream approximation in radiative transfer: Including the angle of the incident radiation. *J. Atmos. Sci.*, 32, 409–418.

Duntley, S. Q., 1942: The optical properties of diffusing materials. *J. Opt. Soc. Amer.*, 32, 61–70.

Emslie, A. G., and J. R. Aronson, 1973: Spectral reflectance and emittance of particulate materials. 1. Theory. *Appl. Opt.*, 12, 2563–2572.

Hansen, J. E., 1969: Exact and approximate solutions for multiple scattering by cloudy and hazy planetary atmospheres. *J. Atmos. Sci.*, 26, 478–487.

Irvine, W. M., 1968: Multiple-scattering by large particles. II. Optically thick layers. *Astrophys. J.*, 152, 823–834.

——, 1975: Multiple scattering in planetary atmospheres. *Icarus*, 25, 175–204.

——, and J. Lenoble, 1974: Solving multiple scattering problems in planetary atmospheres. *Proc. UCLA Int. Conf. Radiation Remote Probing of the Atmosphere*, J. G. Kurujan, Ed., Western Periodicals Co., 1–57.

Joseph, J. H., W. J. Wiscombe and J. A. Weinman, 1976: The delta-Eddington approximation for radiative flux transfer. *J. Atmos. Sci.*, 33, 2452–2459.

Kawata, Y., and W. M. Irvine, 1970: The Eddington approximation for planetary atmospheres. *Astrophys. J.*, 160, 787–790.

Kortüm, G., 1969: *Reflectance Spectroscopy.* Springer-Verlag, 366 pp.

Liou, K. N., 1973: A numerical experiment on Chandrasekhar's discrete-ordinate method for radiative transfer: Application to cloudy and hazy atmospheres. *J. Atmos. Sci.*, 30, 1303–1326.

——, 1974: Analytic two-stream and four-stream solutions for radiative transfer. *J. Atmos. Sci.*, 31, 1473–1475.

Lyzenga, D. R., 1973: Note on the modified two-stream approximation of Sagan and Pollack. *Icarus*, 19, 240–243.

Mudgett, P. S., and L. W. Richards, 1971: Multiple scattering calculations for technology. *Appl. Opt.*, 10, 1485–1502.

——, and ——, 1972: Multiple scattering for technology II. *J. Colloid Interface Sci.*, 39, 551–567.

Rasool, S. I., and S. H. Schneider, 1971: Atmospheric carbon dioxide and aerosols: Effects of large increases on global climate. *Science*, 173, 138–141.

Sagan, C., and J. B. Pollack, 1967: Anisotropic nonconservative scattering and the clouds of Venus. *J. Geophys. Res.*, 72, 469–477.

Shettle, E. P., and J. A. Weinman, 1970: The transfer of solar irradiance through inhomogeneous turbid atmospheres evaluated by Eddington's approximation. *J. Atmos. Sci.*, 27, 1048–1055.

Temkin, R. L., B. C. Weare and F. M. Snell, 1975: Feedback coupling of absorbed solar radiation by three model atmospheres with clouds. *J. Atmos. Sci.*, 32, 873–880.

Van de Hulst, H. C., 1968: Asymptotic fitting, a method for solving anisotropic transfer problems in thick layers. *J. Comput. Phys.*, 3, 291–306.

Weare, B. C., and F. M. Snell, 1974: A diffuse thin cloud atmospheric structure as a feedback mechanism in global climatic modeling. *J. Atmos. Sci.*, 31, 1725–1734.