2 RC Circuits in Time Domain

2.0.1 Capacitors

Capacitors typically consist of two electrodes separated by a non-conducting gap. The quantitiy capacitance C is related to the charge on the electrodes ($+Q$ on one and $-Q$ on the other) and the voltage difference across the capacitor by

$$C = Q/V_C$$

Capacitance is a purely geometric quantity. For example, for two planar parallel electrodes each of area A and separated by a vacuum gap d, the capacitance is (ignoring fringe fields) $C = \epsilon_0 A/d$, where ϵ_0 is the permittivity of vacuum. If a dielectric having dielectric constant κ is placed in the gap, then $\epsilon_0 \rightarrow \kappa\epsilon_0 \equiv \epsilon$. The SI unit of capacitance is the Farad. Typical laboratory capacitors range from $\sim$ 1pF to $\sim$ 1μF.

For DC voltages, no current passes through a capacitor. It "blocks DC". When a time varying potential is applied, we can differentiate our defining expression above to get

$$I = C\frac{dV_C}{dt} \tag{1}$$

for the current passing through the capacitor.

2.0.2 A Basic RC Circuit

Consider the basic RC circuit in Fig. 7. We will start by assuming that $V_{\rm in}$ is a DC voltage source (*e.g.* a battery) and the time variation is introduced by the closing of a switch at time $t = 0$. We wish to solve for $V_{\rm out}$ as a function of time.

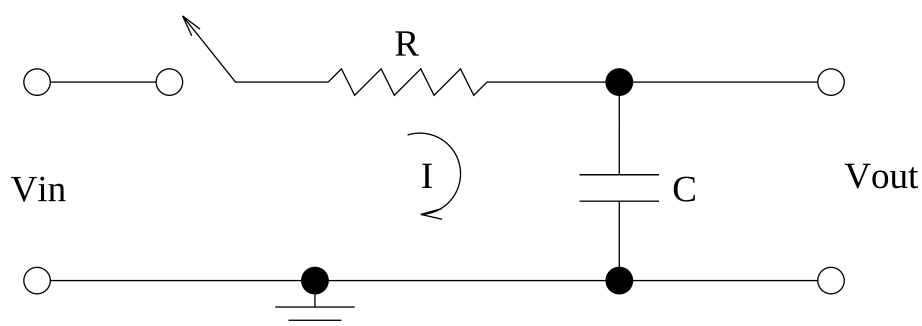

Figure 7: RC circuit — integrator.

Applying Ohm's Law across R gives $V_{\rm in} - V_{\rm out} = IR$. The same current I passes through the capacitor according to $I = C(dV/dt)$. Substituting and rearranging gives (let $V \equiv V_C = V_{\rm out}$):

$$\frac{dV}{dt} + \frac{1}{RC}V = \frac{1}{RC}V_{\rm in} \tag{2}$$

The homogeneous solution is $V = Ae^{-t/RC}$, where A is a constant, and a particular solution is $V = V_{\rm in}$. The initial condition $V(0) = 0$ determines A, and we find the solution

$$V(t) = V_{\rm in}\left[1 - e^{-t/RC}\right] \tag{3}$$

This is the usual capacitor "charge up" solution.

Similarly, a capacitor with a voltage V_i across it which is discharged through a resistor to ground starting at $t = 0$ (for example by closing a switch) can in similar fashion be found to obey

$$V(t) = V_i e^{-t/RC}$$

2.0.3 The "RC Time"

In both cases above, the rate of charge/discharge is determined by the product RC which has the dimensions of time. This can be measured in the lab as the time during charge-up or discharge that the voltage comes to within $1/e$ of its asymptotic value. So in our charge-up example, Equation 3, this would correspond to the time required for V_{out} to rise from zero to 63% of V_{in}.

2.0.4 RC Integrator

From Equation 2, we see that if $V_{\text{out}} \ll V_{\text{in}}$ then the solution to our RC circuit becomes

$$V_{\text{out}} = \frac{1}{RC} \int V_{\text{in}}(t)dt \tag{4}$$

Note that in this case V_{in} can be any function of time. Also note from our solution Eqn. 3 that the limit $V_{\text{out}} \ll V_{\text{in}}$ corresponds roughly to $t \ll RC$. Within this approximation, we see clearly from Eqn. 4 why the circuit above is sometimes called an "integrator".

2.0.5 RC Differentiator

Let's rearrange our RC circuit as shown in Fig. 8.

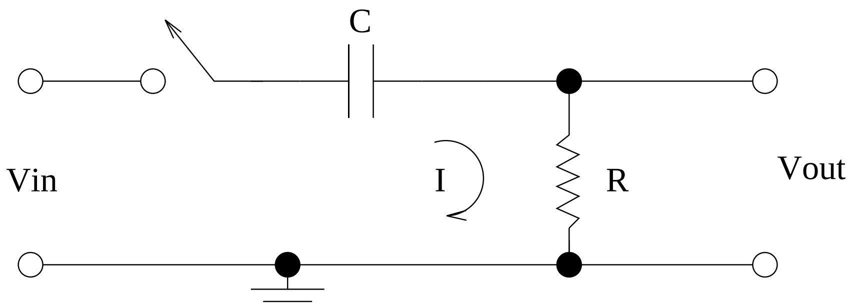

Figure 8: RC circuit — differentiator.

Applying Kirchoff's second Law, we have $V_{\text{in}} = V_C + V_R$, where we identify $V_R = V_{\text{out}}$. By Ohm's Law, $V_R = IR$, where $I = C(dV_C/dt)$ by Eqn. 1. Putting this together gives

$$V_{\text{out}} = RC\frac{d}{dt}(V_{\text{in}} - V_{\text{out}})$$

In the limit $V_{\text{in}} \gg V_{\text{out}}$, we have a differentiator:

$$V_{\text{out}} = RC\frac{dV_{\text{in}}}{dt}$$

By a similar analysis to that of Section 2.0.4, we would see the limit of validity is the opposite of the integrator, *i.e.* $t \gg RC$.

3 Circuit Analysis in Frequency Domain

We now need to turn to the analysis of passive circuits (involving EMFs, resistors, capacitors, and inductors) in frequency domain. Using the technique of the complex impedance, we will be able to analyze time-dependent circuits algebraically, rather than by solving differential equations. We will start by reviewing complex algebra and setting some notational conventions. It will probably not be particularly useful to use the text for this discussion, and it could lead to more confusion. Skimming the text and noting results might be useful.

3.1 Complex Algebra and Notation

Let $\tilde{V}$ be the complex representation of V. Then we can write

$$\tilde{V} = \Re(\tilde{V}) + \imath\Im(\tilde{V}) = Ve^{\imath\theta} = V\left[\cos\theta + \imath\sin\theta\right]$$

where $\imath = \sqrt{-1}$. V is the (real) amplitude:

$$V = \sqrt{\tilde{V}\tilde{V}^*} = \left[\Re^2(\tilde{V}) + \Im^2(\tilde{V})\right]^{1/2}$$

where $*$ denotes complex conjugation. The operation of determining the amplitude of a complex quantity is called taking the *modulus*. The phase θ is

$$\theta = \tan^{-1}\left[\Im(\tilde{V})/\Re(\tilde{V})\right]$$

So for a numerical example, let a voltage have a real part of 5 volts and an imaginary part of 3 volts. Then $\tilde{V} = 5 + 3\imath = \sqrt{34}e^{\imath\tan^{-1}(3/5)}$.

Note that we write the amplitude of $\tilde{V}$, formed by taking its modulus, simply as V. It is often written $|\tilde{V}|$. We will also use this notation if there might be confusion in some context. Since the amplitude will in general be frequency dependent, it will also be written as $V(\omega)$. We will most often be interested in results expressed as amplitudes, although we will also look at the phase.

3.2 Complex Voltage and Current

We want to develop a general technique applicable to any time-dependent signal. Our technique is essentially that of the Fourier transform, although we will not need to actually invoke that formalism. We will analyze our circuits using a single Fourier frequency component, $\omega = 2\pi f$, using for example an AC voltage given by $V(t) = V_0\cos(\omega t + \phi)$ where ϕ is some additional phase to make our AC voltage function completely general. The absolute phase of an AC signal is always arbitrary, only phase differences are important, so we will often drop ϕ from the specification of an input voltage. Analyzing only cosine functions may seem like a cheat, but this is in fact completely general, of course, as we can add (or integrate) over a range of frequencies if need be to recover some particular waveform in the time domain.

It turns out when working with AC currents and voltages that it is much easier to do the algebra using complex numbers. Let our complex Fourier components of voltage and current for a specific ω be written as $\tilde{V} = V_0 e^{\imath(\omega t+\phi_1)}$ and $\tilde{I} = I_0 e^{\imath(\omega t+\phi_2)}$. If we want to know what the actual voltage or current at some point in a circuit might be, we only need to take the real part of this complex function: $V(t) = \Re(\tilde{V}) = V_0 \cos(\omega t + \phi_1)$. Note that our book uses a couple of engineering conventions which most physicists find terribly confusing. First, they use $\jmath = \sqrt{-1}$ instead of $\imath = \sqrt{-1}$. The argument is that $\imath$ gets confused with the current, although for physicists, $\jmath$ is just as confusing. We will stick with $\imath = \sqrt{-1}$. Second, the complex voltage function $\tilde{V}$ defined in the book drops a term of $e^{\imath\omega t}$. This makes the notation slightly more compact, but you have to put this term back in when converting to a real voltage which makes this really kludgy. If you ever find the book referring to a complex AC voltage or current and there seems to be a factor of $e^{\imath\omega t}$ missing, it probably is. We will try to always include it in our discussions.

When first presented, this complex voltage and current notation always seems like a huge and unnecessary complication. As we will shortly see, however, it makes for an almost magical simplification in the analysis of linear circuits composed of inductors, resistors, and capacitors. If you feel uncomfortable with complex numbers, it would be well worth spending a little time reviewing the subject.

3.3 Ohm's Law Generalized

Now, we wish to generalize Ohm's Law by replacing $V = IR$ by $\tilde{V} = \tilde{I}\tilde{Z}$, where $\tilde{Z}$ is the (complex) impedance of a circuit element. Let's see if this can work. We already know that a resistor R takes this form. What about capacitors and inductors?

Our expression for the current through a capacitor, $I = C(dV/dt)$ becomes

$$\tilde{I} = C\frac{d}{dt}Ve^{\imath(\omega t+\phi_1)} = \imath\omega C\tilde{V}$$

Thus, we have an expression of the form $\tilde{V} = \tilde{I}\tilde{Z}_C$ for the impedance of a capacitor, $\tilde{Z}_C$, if we make the identification $\tilde{Z}_C = 1/(\imath\omega C)$.

For an inductor of self-inductance L, the voltage *drop* across the inductor is given by Lenz's Law: $V = L(dI/dt)$. (Note that the voltage drop has the opposite sign of the induced EMF, which is usually how Lenz's Law is expressed.) Our complex generalization leads to

$$\tilde{V} = L\frac{d}{dt}\tilde{I} = L\frac{d}{dt}Ie^{\imath(\omega t+\phi_2)} = \imath\omega L\tilde{I}$$

So again the form of Ohm's Law is satisfied if we make the identification $\tilde{Z}_L = \imath\omega L$.

To summarize our results, Ohm's Law in the complex form $\tilde{V} = \tilde{I}\tilde{Z}$ can be used to analyze circuits which include resistors, capacitors, and inductors if we use the following:

- resistor of resistance R: $\tilde{Z}_R = R$
- capacitor of capacitance C: $\tilde{Z}_C = 1/(\imath\omega C) = -\imath/(\omega C)$
- inductor of self-inductance L: $\tilde{Z}_L = \imath\omega L$

A word on nomenclature. Impedance is the complex generalization of resistance which is used in the complex representation of Ohm's law. The real component of a complex impedance is the resistance, while the imaginary part is called the *reactance*. Inductance and capacitance are two different types of reactance. Even with all of this complication, resistors, capacitors, and inductors are all *linear* elements which can only change the amplitude and phase of a single Fourier component.

3.3.1 Capacitor Impedance

If we wanted to know the current flowing through a capacitor driven by an AC voltage, we can now simply use our generalized, complex Ohm's law to give us an answer. Since $\tilde{I} = \tilde{V}/\tilde{Z}$ and $\tilde{Z}_C = 1/(\imath\omega C)$, we have $\tilde{I} = \imath\omega C\tilde{V}$. To get the real current, we need to take the real part of the right-hand side:

$$I(t) = \Re\left[\imath\omega CV_0e^{\imath\omega t}\right] = \Re\left[\imath\omega CV_0(\cos\omega t + \imath\sin\omega t)\right] = -\omega CV_0\sin\omega t.$$

So for $V(t) = V_0\cos\omega t$ we end up with $I(t) = -\omega CV_0\sin\omega t$, which leads the voltage by a phase of 90 degrees and increases by the frequency-dependent factor of ωC. A very good way to intuitively think about capacitors, in fact, is as a device with a frequency-dependent impedance. High frequency components pass through a capacitor, while low frequency components are blocked.

3.3.2 Combining Impedances

It is significant to point out that because the algebraic form of Ohm's Law is preserved, impedances follow the same rules for combination in series and parallel as we obtained for resistors previously. So, for example, two capacitors in parallel would have an equivalent impedance given by $1/\tilde{Z}_p = 1/\tilde{Z}_1 + 1/\tilde{Z}_2$. Using our definition $\tilde{Z}_C = -\imath/\omega C$, we then recover the familiar expression $C_p = C_1 + C_2$. So we have for any two impedances in series (clearly generalizing to more than two):

$$\tilde{Z}_s = \tilde{Z}_1 + \tilde{Z}_2$$

And for two impedances in parallel:

$$\tilde{Z}_p = \left[1/\tilde{Z}_1 + 1/\tilde{Z}_2\right]^{-1} = \frac{\tilde{Z}_1\tilde{Z}_2}{\tilde{Z}_1 + \tilde{Z}_2}$$

And, accordingly, our result for a voltage divider generalizes (see Fig. 9) to

$$\tilde{V}_{\text{out}} = \tilde{V}_{\text{in}}\left[\frac{\tilde{Z}_2}{\tilde{Z}_1 + \tilde{Z}_2}\right] \tag{5}$$

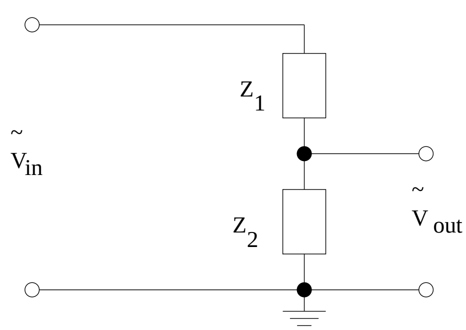

Figure 9: The voltage divider generalized.

Now we are ready to apply this technique to some examples.

3.4 A High-Pass RC Filter

The configuration we wish to analyze is shown in Fig. 10. Note that it is the same as Fig. 7 of the notes. However, this time we apply a voltage which is sinusoidal: $\tilde{V}_{\rm in}(t) = V_{\rm in}e^{\imath(\omega t+\phi)}$. As an example of another common variation in notation, the figure indicates that the input is sinusoidal ("AC") by using the symbol shown for the input. Note also that the input and output voltages are represented in the figure only by their amplitudes $V_{\rm in}$ and $V_{\rm out}$, which also is common. This is fine, since the method we are using to analyze the circuit (complex impedances) shouldn't necessarily enter into how we describe the physical circuit.

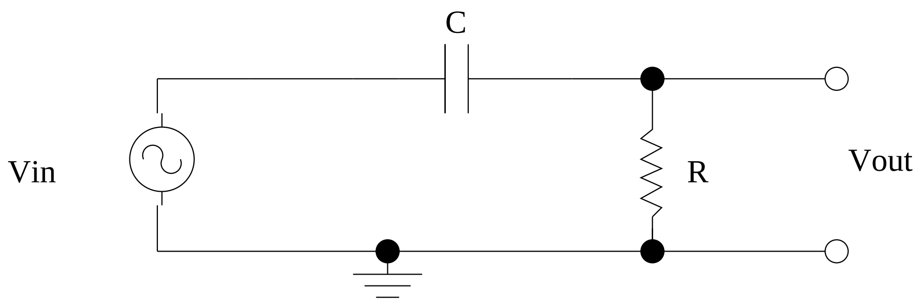

Figure 10: A high-pass filter.

We see that we have a generalized voltage divider of the form discussed in the previous section. Therefore, from Eqn. 5 we can write down the result if we substitute $\tilde{Z}_1 = \tilde{Z}_C = -\imath/(\omega C)$ and $\tilde{Z}_2 = \tilde{Z}_R = R$:

$$\tilde{V}_{\rm out} = \tilde{V}_{\rm in}\left[\frac{R}{R - \imath/(\omega C)}\right]$$

At this point our result is general, and includes both amplitude and phase information. Often, we are only interested in amplitudes. We can divide by $\tilde{V}_{\rm in}$ on both sides and find the amplitude of this ratio (by multiplying by the complex conjugate then taking the square root). The result is often referred to as the *transfer function* of the circuit, which we can designate by $T(\omega)$.

$$T(\omega) \equiv \frac{|\tilde{V}_{\rm out}|}{|\tilde{V}_{\rm in}|} = \frac{V_{\rm out}}{V_{\rm in}} = \frac{\omega RC}{[1 + (\omega RC)^2]^{1/2}} \tag{6}$$

Examine the behavior of this function. Its maximum value is one and minimum is zero. You should convince yourself that this circuit attenuates low frequencies and "passes" (transmits with little attenuation) high frequencies, hence the term *high-pass filter*. The cutoff between high and low frequencies is conventionally described as the frequency at which the transfer function is $1/\sqrt{2}$. This is approximately equal to an attenuation of 3 *decibels*, which is a description often used in engineering (see below). From Eqn. 6 we see that $T = 1/\sqrt{2}$ occurs at a frequency

$$2\pi f_{\rm 3db} = \omega_{\rm 3db} = 1/(RC) \tag{7}$$

The decibel scale works as follows: db$= 20\log_{10}(A_1/A_2)$, where A_1 and A_2 represent any real quantity, but usually are amplitudes. So a ratio of 10 corresponds to 20 db, a ratio of 2 corresponds to 6 db, $\sqrt{2}$ is approximately 3 db, *etc.*

3.5 A Low-Pass RC Filter

An analogy with the analysis above, we can analyze a low-pass filter, as shown in Fig. 11.

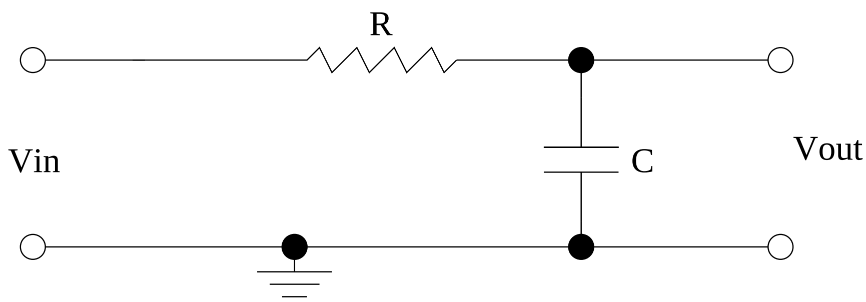

Figure 11: A low-pass filter.

You should find the following result for the transfer function:

$$T(\omega) \equiv \frac{|\tilde{V}_{\mathrm{out}}|}{|\tilde{V}_{\mathrm{in}}|} = \frac{V_{\mathrm{out}}}{V_{\mathrm{in}}} = \frac{1}{[1+(\omega RC)^2]^{1/2}} \tag{8}$$

You should verify that this indeed exhibits "low pass" behavior. And that the 3 db frequency is the same as we found for the high-pass filter:

$$2\pi f_{3\mathrm{db}} = \omega_{3\mathrm{db}} = 1/(RC) \tag{9}$$

We note that the two circuits above are equivalent to the circuits we called "differentiator" and "integrator" in Section 2. However, the concept of high-pass and low-pass filters is much more general, as it does not rely on an approximation.

An aside. One can compare our results for the RC circuit using the complex impedance technique with what one would obtain by starting with the differential equation (in time) for an RC circuit we obtained in Section 2, taking the Fourier transform of that equation, then solving (algebraically) for the transform of V_{out}. It should be the same as our result for the amplitude V_{out} using impedances. After all, that is what the impedance technique is doing: transforming our time-domain formuation to one in frequency domain, which, because of the possibility of analysis using a single Fourier frequency component, is particularly simple.

3.6 Bode Plots

Equation 8 describes the frequency-dependent attenuation of a low-pass filter. A common way to visualize this information is by plotting the transfer function (in db) versus the frequency on a log scale. The resulting log-log plot, called a Bode plot, is shown in Figure 3.6.

The main features of a low-pass filter are clearly visible. At $\omega = \omega_{3\mathrm{db}} = 1/(RC)$, the attenuation of this circuit it 3 db, or $1/\sqrt{2}$. At frequencies well below the 3 db point, the attenuation is minimal, while at frequencies above the 3 db point, the attenuation falls linearly (on a log-log plot) at a rate of 20 db per decade.

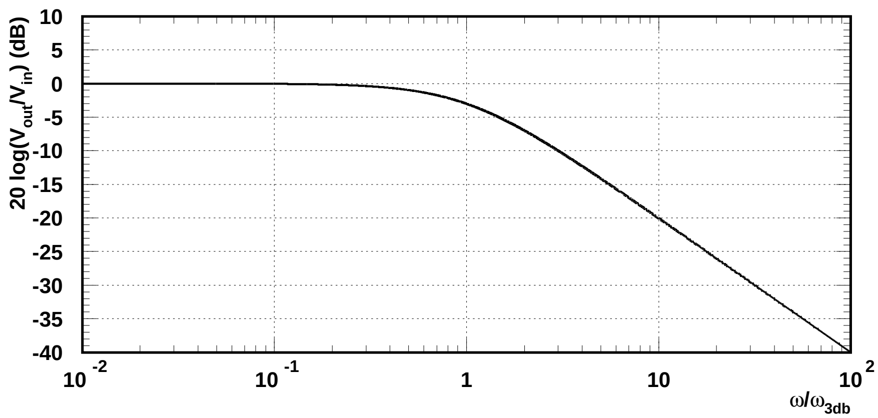

Figure 12: Bode plot of a low-pass filter.

3.7 Frequency Domain Analysis (contd.)

Before we look at some more examples using our technique of complex impedance, let's look at some related general concepts.

3.7.1 Reactance

First, just a redefintion of what we already have learned. The term *reactance* is often used in place of impedance for capacitors and inductors. Reviewing our definitions of impedances from Section 3.3 we define the reactance of a capacitor $\tilde{X}_C$ to just be equal to its impedance: $\tilde{X}_C \equiv -\imath/(\omega C)$. Similarly, for an inductor $\tilde{X}_L \equiv \imath\omega L$. This is the notation used in the text.

However, an alternative but common useage is to define the reactances as real quantities. This is done simply by dropping the $\imath$ from the definitions above. The various reactances present in a circuit can by combined to form a single quantity X, which is then equal to the imaginary part of the impedance. So, for example a circuit with R, L, and C in series would have total impedance

$$\tilde{Z} = R + \imath X = R + \imath(X_L + X_C) = R + \imath(\omega L - \frac{1}{\omega C})$$

A circuit which is "reactive" is one for which X is non-negligible compared with R.

3.7.2 General Solution

As stated before, our technique involves solving for a single Fourier frequency component such as $\tilde{V} = Ve^{\imath(\omega t+\phi)}$. You may wonder how our results generalize to other frequencies and to input waveforms other than pure sine waves. The answer in words is that we Fourier decompose the input and then use these decomposition amplitudes to weight the output we found for a single frequency, $V_{\rm out}$. We can formalize this within the context of the Fourier transform, whch will also allow us to see how our time-domain differential equation became transformed to an algebraic equation in frequency domain.

Consider the example of the RC low-pass filter, or integrator, circuit of Fig. 7. We obtained the differential equation given by Eq. 2. We wish to take the Fourier transform of this equation. Define the Fourier transform of $V(t)$ as

$$v(\omega) \equiv \mathcal{F}\{V(t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt e^{-\imath\omega t} V(t) \tag{10}$$

Recall that $\mathcal{F}\{dV/dt\} = \imath\omega\mathcal{F}\{V\}$. Therefore our differential equation becomes

$$\imath\omega v(\omega) + v(\omega)/(RC) = \mathcal{F}\{V_{\rm in}(t)\}/(RC) \tag{11}$$

Solving for $v(\omega)$ gives

$$v(\omega) = \frac{\mathcal{F}\{V_{\rm in}(t)\}}{1 + \imath\omega RC} \tag{12}$$

The general solution is then the real part of the inverse Fourier transform:

$$\tilde{V}(t) = \mathcal{F}^{-1}\{v(\omega)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega' e^{\imath\omega' t} v(\omega') \tag{13}$$

In the specific case we have considered so far of a single Fourier component of frequency ω, *i.e.* $\tilde{V}_{\rm in} = V_i e^{\imath\omega t}$, then $\mathcal{F}\{\tilde{V}_{\rm in}(t)\} = \sqrt{2\pi}\delta(\omega - \omega')$, and we recover our previous result for the transfer function:

$$\tilde{T} = \tilde{V}/\tilde{V}_{\rm in} = \frac{1}{1 + \imath\omega RC} \tag{14}$$

For an arbitrary functional form for $V_{\rm in}(t)$, one could use Eqns. 12 and 13. Note that one would go through the same steps if $V_{\rm in}(t)$ were written as a Fourier series rather than a Fourier integral. Note also that the procedure carried out to give Eqn. 11 is formally equivalent to our use of the complex impedances: In both cases the differential equation is converted to an algebraic equation.

3.8 Phase Shift

We now need to discuss finding the phase ϕ of our solution. To do this, we proceed as previously, for example like the high-pass filter, but this time we preserve the phase information by not taking the modulus of $\tilde{V}_{\rm out}$. The input to a circuit has the form $\tilde{V}_{\rm in} = V_{\rm in} e^{\imath(\omega t + \phi_1)}$, and the output $\tilde{V}_{\rm out} = V_{\rm out} e^{\imath(\omega t + \phi_2)}$. We are usually only interested in the phase difference $\phi_2 - \phi_1$ between input and output, so, for convenience, we can choose $\phi_1 = 0$ and set the *phase shift* to be $\phi_2 \equiv \phi$. Physically, we must choose the real or imaginary part of these expressions. Conventionally, the real part is used. So we have:

$$V_{\rm in}(t) = \Re(\tilde{V}_{\rm in}) = V_{\rm in}(\omega)\cos(\omega t)$$

and

$$V_{\rm out}(t) = \Re(\tilde{V}_{\rm out}) = V_{\rm out}(\omega)\cos(\omega t + \phi)$$

Let's return to our example of the high-pass filter to see how to calculate the phase shift. We rewrite the expression from Section 3.4 and then multiply numerator and denominator by the complex conjugate of the denominator:

$$\tilde{V}_{\rm out} = \tilde{V}_{\rm in}\left[\frac{R}{R - \imath/(\omega C)}\right] = V_{\rm in} e^{\imath\omega t} \frac{1 + \imath/(\omega RC)}{1 + 1/(\omega RC)^2}$$

By recalling the general form $a + \imath b = \sqrt{a^2+b^2}\, e^{\imath\phi}$, where $\phi = \tan^{-1}(b/a)$, we can write

$$1 + \imath/(\omega RC) = \left[1 + \left(\frac{1}{\omega RC}\right)^2\right]^{1/2} e^{\imath\phi}$$

allowing us to read off the phase shift:

$$\phi = \tan^{-1}\left(1/(\omega RC)\right) \tag{15}$$

Our solution for $\tilde{V}_{\rm out}$ is then

$$\tilde{V}_{\rm out} = \frac{V_{\rm in} e^{\imath\omega t+\phi}}{\left[1 + (\frac{1}{\omega RC})^2\right]^{1/2}}$$

This, of course, yields the same $|\tilde{V}_{\rm out}|$ as we found before in Eqn. 6 of Section 3.4. But now we also have included the phase information. The "real" time-dependent solution is then just the real part of this:

$$V_{\rm out}(t) = \Re(\tilde{V}_{\rm out}) = V_{\rm out}\cos(\omega t + \phi)$$

where ϕ is given by Eqn. 15.

3.9 Power in Reactive Circuits

Recall that for DC voltages and currents the power associated with a circuit element carrying current I with voltage change V is just $P = VI$. Now, for time-varying voltages and currents we have to be more careful. We could still define an instantaneous power as the product $V(t)I(t)$. However, it is generally more useful to average the power over time.

3.9.1 General Result for AC

Since we are considering Fourier components, we will average the results over one period $T = 1/f = 2\pi/\omega$. Therefore, the time-averaged power is

$$<P>= \frac{1}{T}\int_0^T V(t)I(t)dt$$

where the brackets indicate the time average. Let the voltage and current be out of phase by an arbitrary phase angle ϕ. So we have $V(t) = V_0\cos(\omega t)$ and $I(t) = I_0\cos(\omega t + \phi)$. We can plug these into the expression for $<P>$ and simplify using the following: $\cos(\omega t + \phi) = \cos(\omega t)\cos(\phi) - \sin(\omega t)\sin(\phi)$; $\int_0^T \sin(\omega t)\cos(\omega t)dt = 0$; and $(1/T)\int_0^T \sin^2(\omega t)dt = (1/T)\int_0^T \cos^2(\omega t)dt = 1/2$. This yields

$$<P>= \frac{1}{2}V_0 I_0\cos\phi = V_{\rm RMS} I_{\rm RMS}\cos\phi \tag{16}$$

In the latter expression we have used the "root mean squared", or *RMS*, amplitudes. Using voltage as an example, the RMS and standard amplitudes are related by

$$V_{\rm RMS} \equiv \left[\frac{1}{T}\int_0^T V^2(t)dt\right]^{1/2} = \left[\frac{1}{T}\int_0^T V_0^2\cos^2(\omega t)dt\right]^{1/2} = V_0/\sqrt{2} \tag{17}$$

3.9.2 Power Using Complex Quantities

Our results above can be simply expressed in terms of $\tilde{V}$ and $\tilde{I}$. Equivalent to above, we start with $\tilde{V}(t) = V_0 e^{\imath\omega t}$ and $\tilde{I}(t) = I_0 e^{\imath(\omega t+\phi)}$. By noting that

$$\Re(\tilde{V}^* \tilde{I}) = \Re\left(V_0 I_0 (\cos\phi + \imath \sin\phi)\right) = V_0 I_0 \cos\phi$$

we identify an expression for average power which is equivalent to Eqn. 16 :

$$< P >= \frac{1}{2}\Re(\tilde{V}^* \tilde{I}) = \frac{1}{2}\Re(\tilde{V} \tilde{I}^*) \tag{18}$$

3.10 An RLC Circuit Example

We can apply our technique of impedance to increasingly more intricate examples, with no more effort than a commensurate increase in the amount of algebra. The RLC circuit of Fig. 13 exemplifies some new qualitative behavior.

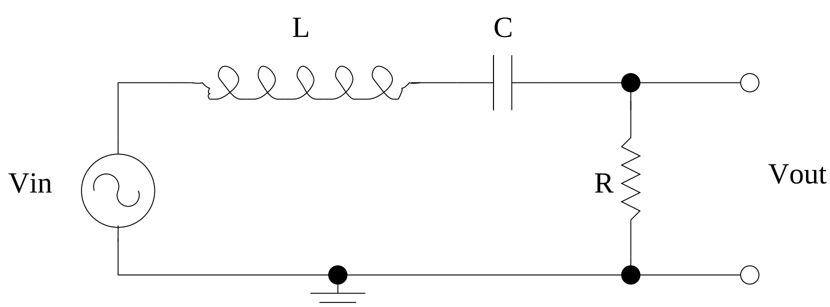

Figure 13: A RLC circuit. Several filter types are possible depending upon how V_{out} is chosen. In the case shown, the circuit gives a resonant output.

We can again calculate the output using our generalized voltage divider result of Eqn. 5. In this case, the $\tilde{Z}_1$ consists of the inductor and capacitor in series, and $\tilde{Z}_2$ is simply R. So,

$$\tilde{Z}_1 = \imath\omega L - \imath/(\omega C) = \frac{\imath L}{\omega}\left(\omega^2 - \omega_0^2\right)$$

where we have defined the *LC resonant frequency* $\omega_0 \equiv 1/\sqrt{LC}$. We then obtain for the transfer function:

$$T(\omega) \equiv \frac{|\tilde{V}_{\text{out}}|}{|\tilde{V}_{\text{in}}|} = \frac{R}{|R + \tilde{Z}_1|} = \frac{\omega\gamma}{\left[\omega^2\gamma^2 + (\omega^2 - \omega_0^2)^2\right]^{1/2}}$$

where $\gamma \equiv R/L$ is the "R-L frequency".

$T(\omega)$ indeed exhibits a resonance at $\omega = \omega_0$. The quality factor Q, defined as the ratio of ω_0 to the width of the resonance is given by $Q \approx \omega_0/(2\gamma)$ for $\gamma \ll \omega_0$. Such circuits have many applications. For example, a high-Q circuit, where $V_{\text{in}}(t)$ is the signal on an antenna, can be used as a receiver.

As was shown in class, we achieve different behavior if we choose to place the output across the capacitor or inductor, rather than across the resistor, as above. Rather than a resonant circuit, choosing $V_{\text{out}} = V_C$ yields a low-pass filter of the form

$$T(\omega) = \frac{|-\imath/(\omega C)|}{|R + \tilde{Z}_1|} = \frac{\omega_0^2}{[\omega^2\gamma^2 + (\omega^2 - \omega_0^2)^2]^{1/2}}$$

The cutoff frequency is ω_0, and for $\omega \gg \omega_0$ then $T \sim \omega^{-2}$ ("12 db per octave"), which more closely approaches ideal step function-like behavior than the RC low pass filter, for which $T \sim \omega^{-1}$ for $\omega \gg \omega_0$ ("6 db per octave"). As you might suspect, choosing $V_{\text{out}} = V_L$ provides a high-pass filter with cutoff at ω_0 and $T \sim \omega^{-2}$ for $\omega \ll \omega_0$.

3.11 More Filters

3.11.1 Combining Filter Sections

Filter circuits can be combined to produce new filters with modified functionality. An example is the homework problem (6) of page 59 of the text, where a high-pass and a low-pass filter are combined to form a "band-pass" filter. As discussed at length in Section 1.5, it is important to design a "stiff" circuit, in which the next circuit element does not load the previous one, by requiring that the output impedance of the first be much smaller than the input impedance of the second. We can standardize this inequality by using a factor of 10 for the ratio $|\tilde{Z}_{\text{in}}|/|\tilde{Z}_{\text{out}}|$.

3.11.2 More Powerful Filters

This technique of cascading filter elements to produce a better filter is discussed in detail in Chapter 5 of the text. In general, the transfer functions of such filters take the form (for the low-pass case):

$$T(\omega) = \left[1 + \alpha_n (f/f_c)^{2n}\right]^{-1/2}$$

where f_c is the 3 db frequency, α_n is a coefficient depending upon the type of filter, and n is the filter "order," often equal to the number of filtering capacitors.

3.11.3 Active Filters

Filters involving LC circuits are very good, better than the simple RC filters, as discussed above. Unfortunately, inductors are, in practice, not ideal lumped circuit elements and are difficult to fabricate. In addition, filters made entirely from passive elements tend to have a lot of attenuation. For these reasons active filters are most commonly used where good filtering is required. These typically use operational amplifiers (which we will discuss later), which can be configured to behave like inductors, and can have provide arbitrary voltage gain. Again, this is discussed in some detail in Chapter 5. When we discuss op amps later, we will look at some examples of very simple active filters. At high frequencies (for example RF), op amps fail, and one most fall back on inductors.