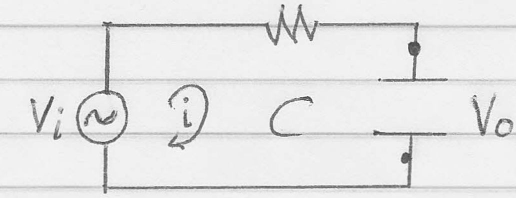
Linear Circuit Analysis

Given:



General Voltage, frequency f

RC circuit, an input voltage source V_i, output V_o of interest.

Find: The transfer function $H(f) = \dfrac{V_o(f)}{V_i(f)}$

Solution: Assume waveforms $V_i(t) = R_e\{\tilde{V}_i(f)\, e^{j\omega t}\}$

$$V_o(t) = R_e\{\tilde{V}_o(f)\, e^{j\omega t}\}$$

Where $j = \sqrt{-1}$, $\omega = 2\pi f$, f = frequency,

$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$

$\tilde{V}_i(f)$ is the complex amplitude of the source waveform,

$$\tilde{V}_i(f) = \overset{Real}{\hat{V}_i(f)} + j\,\overset{Imagi}{\hat{V}_i(f)}$$

Once we know $H(f)$, any V_i and V_o can be figured out using $V_o(t) = \mathcal{F}^{-1}\{\mathcal{F}(V_i(t))H(f)\}$

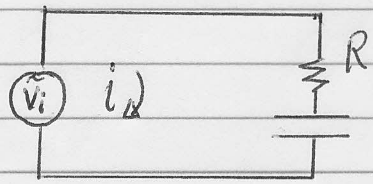
$\mathcal{F}$ is the Fourier transform operator, and its inverse.

By the convolution theorem we could also write

$V_o(t) = V_i(t) \circledast H(t)$ * ← Inverse Fourier transform of $H(f)$

$\circledast$: convolution operator

What happened to Ohms Law?



$Z_c = \frac{-j}{\omega C}$ } $Z = R + \frac{(-j)}{\omega C}$

$\tilde{V}_i = i Z$ → Can calculate i

$i = \frac{V_i}{Z}$

Then $V_R \equiv$ Resistor voltage $= iR$,

$$V_R = \frac{R}{Z} V_i$$

Capacitor voltage $\equiv V_c = \frac{Z_c}{Z} V_i$

This is our normal voltage divider.

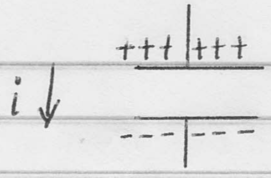
That's some fine mathematics.

Where is the physics?

Here. $i\downarrow$ Resistor. Voltage $= iR$

Power dissipated $= V_R i = i^2 R = \frac{V^2}{R}$

And here:



$C = \frac{Q}{V_c}$ = capacitance

Q = charge on capacitor

V_c = voltage across capacitor.

Suppose we turn on the current i at time $t = 0$.

Later the charge piled up on the capacitor is

$$Q(t) = \int_0^t i(\tau)\,d\tau$$

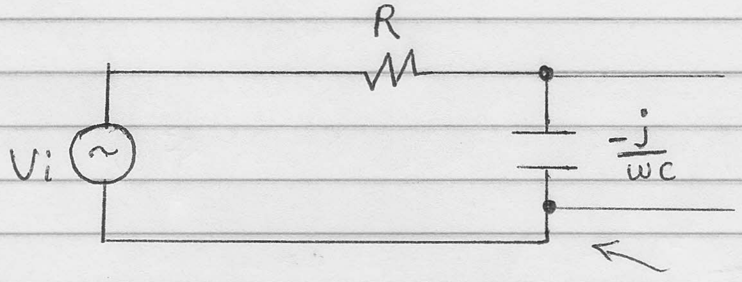
The capacitor has capacity to hold charge!

Remember, $i = \frac{dQ}{dt}$ is current.

The voltage across the capacitor is

$$V_c = \frac{Q}{C} = \frac{\int_0^t i(\tau)\,d\tau}{C}.$$

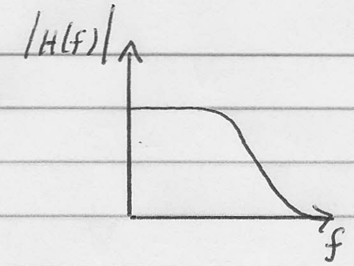
This equation says that with a large capacity C it takes a long time to build up the voltage V_c.



$$Z_c = \frac{-j}{\omega C}$$

This is a low pass filter

Low pass filter



Why? $Z_c(\omega \to 0) = \infty$!

$Z_c(\omega \to \infty) = 0$!

It is a simple voltage divider circuit.

$$Z_c \gg R \text{ as } f \to 0, \quad Z_c \ll R \text{ as } f \to \infty$$

Actual transfer function:

$$H(f) = \frac{Z_c}{Z_c + R} = \frac{1}{1 + \frac{R}{Z_c}} = \frac{1}{1 + j\omega RC}$$

The "RC time constant" is $\underline{\tau \equiv RC}$

$$H(f) = \frac{1}{1 + 2\pi j f \tau}$$

Complex number.

$$\boxed{\begin{aligned} |H(f)| &= \frac{1}{\sqrt{1 + (2\pi f \tau)^2}} \\ \text{Phase } H(f) &= -\tan^{-1}(2\pi f \tau) \end{aligned}}$$ Magnitude

Why a phase shift? It takes time for the capacitor to charge.

At $f \to$ small, e.g., $f\tau << 2\pi$,

$$|H(f)| = 1, \quad \text{phase} = 0$$

At $f \to \infty$, e.g. $f\tau >> 2\pi$,

$$|H(f)| \to 0, \quad \text{phase} \to \frac{\pi}{2}$$

The filter has both an effect on the signal amplitude and phase.

Fourier Analysis:

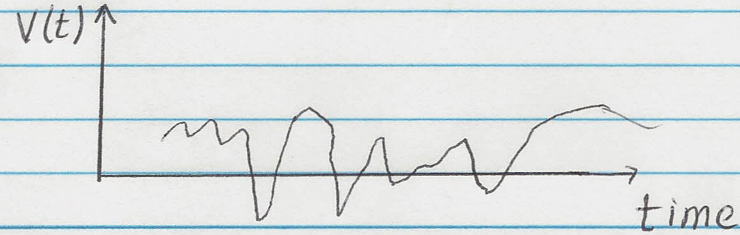
Let $V(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{V}(f) e^{2\pi f t j} df$

$$\hat{V}(f) = \int_{-\infty}^{\infty} V(t) e^{-2\pi f t j} dt.$$

Defines a Fourier transform pair

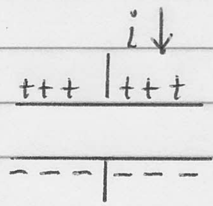
Note: $e^{2\pi f t j} = \cos(2\pi f t) + j\sin(2\pi f t)$

(Euler Identity) a sine and cosine wave



Physically $\hat{V}(f)$ is the superposition of sine and cosine waves needed to reproduce $V(t)$.

Where does $Z_c = \frac{-j}{wC}$ come from?



$$i = \frac{dQ}{dt} \qquad C = \frac{Q}{V}$$

Transform $i(t)$ and $V(t)$ to the frequency domain $\tilde{i}(f)$, $\hat{V}(f)$.

$$\boxed{Z_c \equiv \frac{\hat{V}(f)}{\hat{i}(f)} = \frac{\hat{Q}(f)}{C\,\hat{i}(f)} \quad \ldots (1)}$$

$$Q(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{Q}(f)\, e^{j2\pi f t}\, df$$

$$\frac{dQ}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{Q}(f)\, j2\pi f\, df$$

In other words,

$$i(t) = \frac{dQ}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{Q}(f)\, j2\pi f\, df$$

$$i(t) = \mathcal{F}^{-1}\left\{\hat{Q}(f)\, j2\pi f\right\}$$

Doing $\mathcal{F}\{\ \}$ to both sides,

$$\hat{i}(f) = \hat{Q}(f)\, j2\pi f \quad \ldots \quad (2)$$

Using eq.(2) in eq.(1),

$$\Rightarrow \boxed{Z_C = \frac{-j}{2\pi f C} = \frac{-j}{\omega C}}$$

We are working in the frequency domain when we use Z_C.