Example to illustrate circulation:

Helmholtz theorem states for sufficient well behaved $\vec{U}$,

$\Rightarrow$ Think Velocity field

$$\vec{U} = \underbrace{\vec{\nabla}\phi}_{\text{Scalar Potential (not rotating Part)}} + \underbrace{\vec{\nabla} \times \vec{\Psi}}_{\text{Vector potential (rotating Part)}} \qquad \text{where } \vec{\nabla}\cdot\vec{\Psi} = 0$$

Class

For:

$$\vec{\Psi} = \left(\frac{x^2+y^2}{4}\right)\hat{z}$$

Show: $\vec{\nabla}\cdot\vec{\Psi} = 0 \qquad \nabla^2\vec{\Psi} = 1\,\hat{z}$

$$\vec{\nabla} \times \vec{\Psi} = \hat{x}\frac{y}{2} - \hat{\jmath}\frac{x}{2} \equiv \vec{U}_t$$

For:

$$\phi = (x^2+y^2)/4$$

Show:

$$\vec{\nabla}\phi = x/_2\,\hat{x} + y/_2\,\hat{y} \equiv \vec{U}_e$$

$$\nabla^2\phi = \vec{\nabla}\cdot\vec{\nabla}\phi = 1$$

Thus: $\vec{\nabla}\cdot\vec{U} = \vec{\nabla}\cdot\vec{\nabla}\phi = \nabla^2\phi = 1$

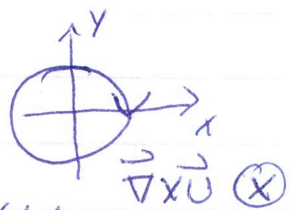
(divergence of $\vec{U} = 1$)

$$\vec{\nabla} \times \vec{U} = \overbrace{\vec{\nabla} \times \vec{\nabla}\phi}^{=0} + \vec{\nabla} \times (\vec{\nabla} \times \vec{\Psi})$$

$$= \vec{\nabla}(\vec{\nabla}\cdot\vec{\Psi}) - \nabla^2\vec{\Psi}$$

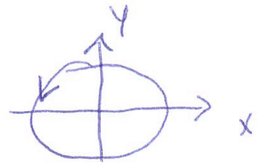
$$= -\hat{z}$$

(Vorticity of $\vec{U} = -1\,\hat{z}$ $\Rightarrow$




(right hand rule)

$A$ = area of loop




$\hat{n} = \hat{z}$

We can now evaluate the circulation:

$$\oint \vec{U} \cdot d\vec{s} = \iint_A \underbrace{(\vec{\nabla} \times \vec{U})}_{-\hat{z}} \cdot \underbrace{\hat{n}}_{\hat{z}} \, dA = -A$$

Graphing $\vec{U}$:

$$\vec{U} = \vec{\nabla}\phi + \vec{\nabla} \times \vec{\Psi} \equiv \vec{U}_e + \vec{U}_t$$

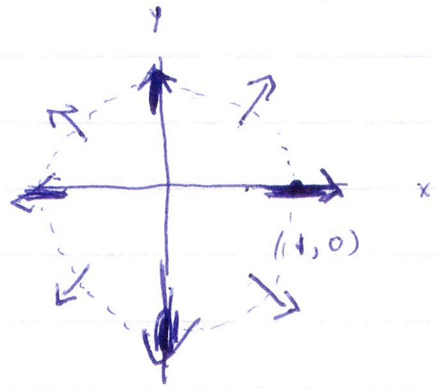
$$\vec{U} = \overbrace{\frac{x}{2}\hat{x} + \frac{y}{2}\hat{y}}^{\vec{U}_e} + \overbrace{\hat{i}\frac{y}{2} - \hat{j}\frac{x}{2}}^{\vec{U}_t}$$

Note:

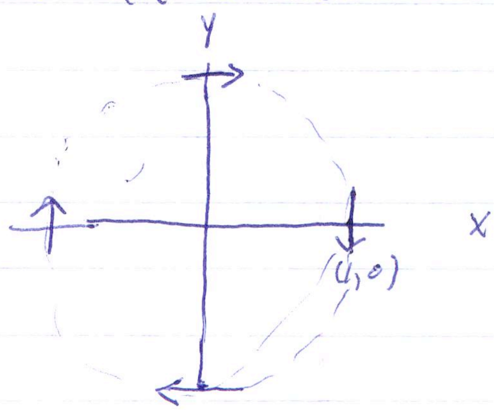
show

$$|\vec{U}_e| = |\vec{U}_t| = \left(\frac{x^2}{4} + \frac{y^2}{4}\right)^{1/2}$$

Contours of equal magnitude are circles




$\vec{U}_e$

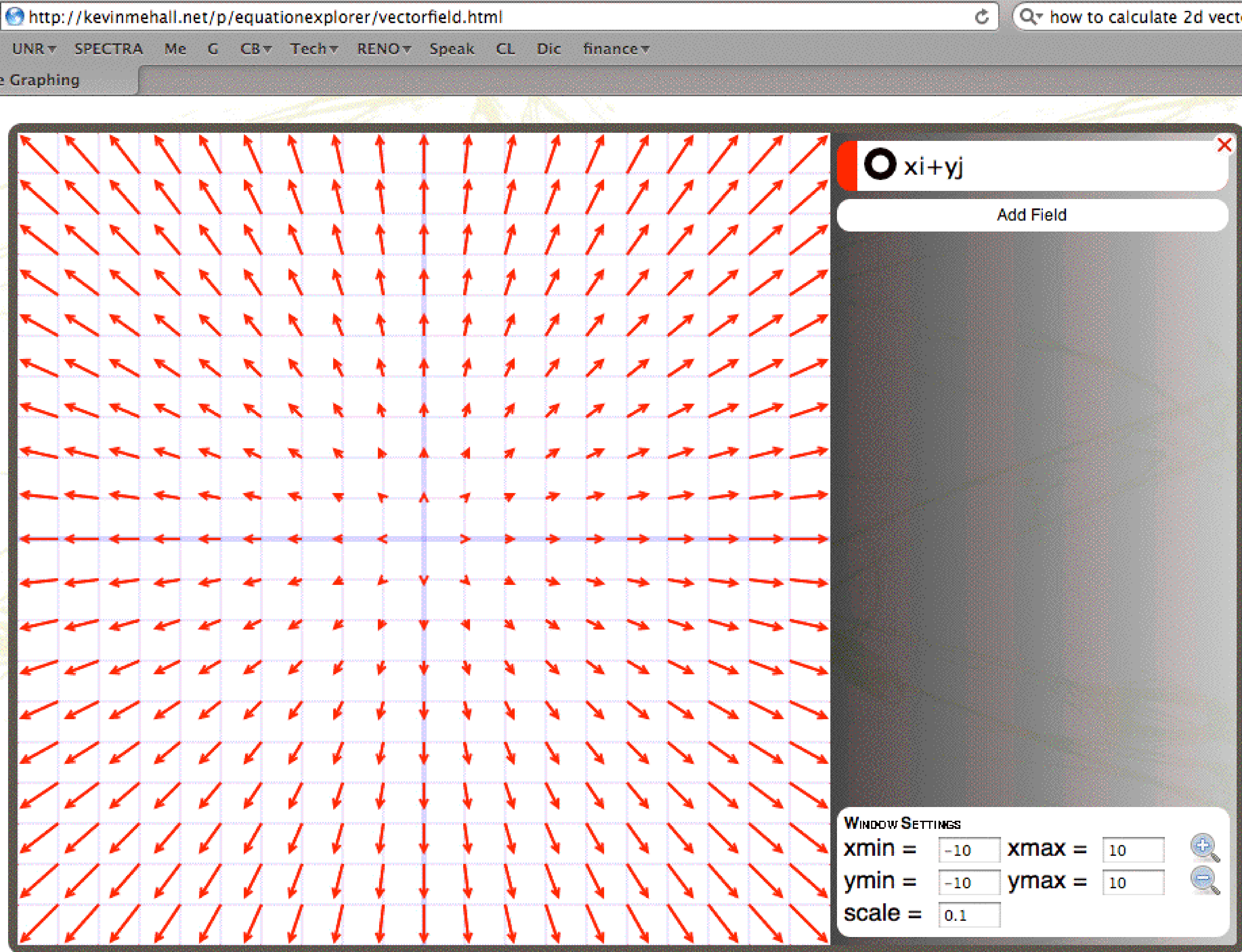



$\vec{U}_t$

Negative Vorticity
(rotates in clockwise sense)

Can use online facility to plot
Very nicely

Kevin Mchall Online Graphing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Add Field
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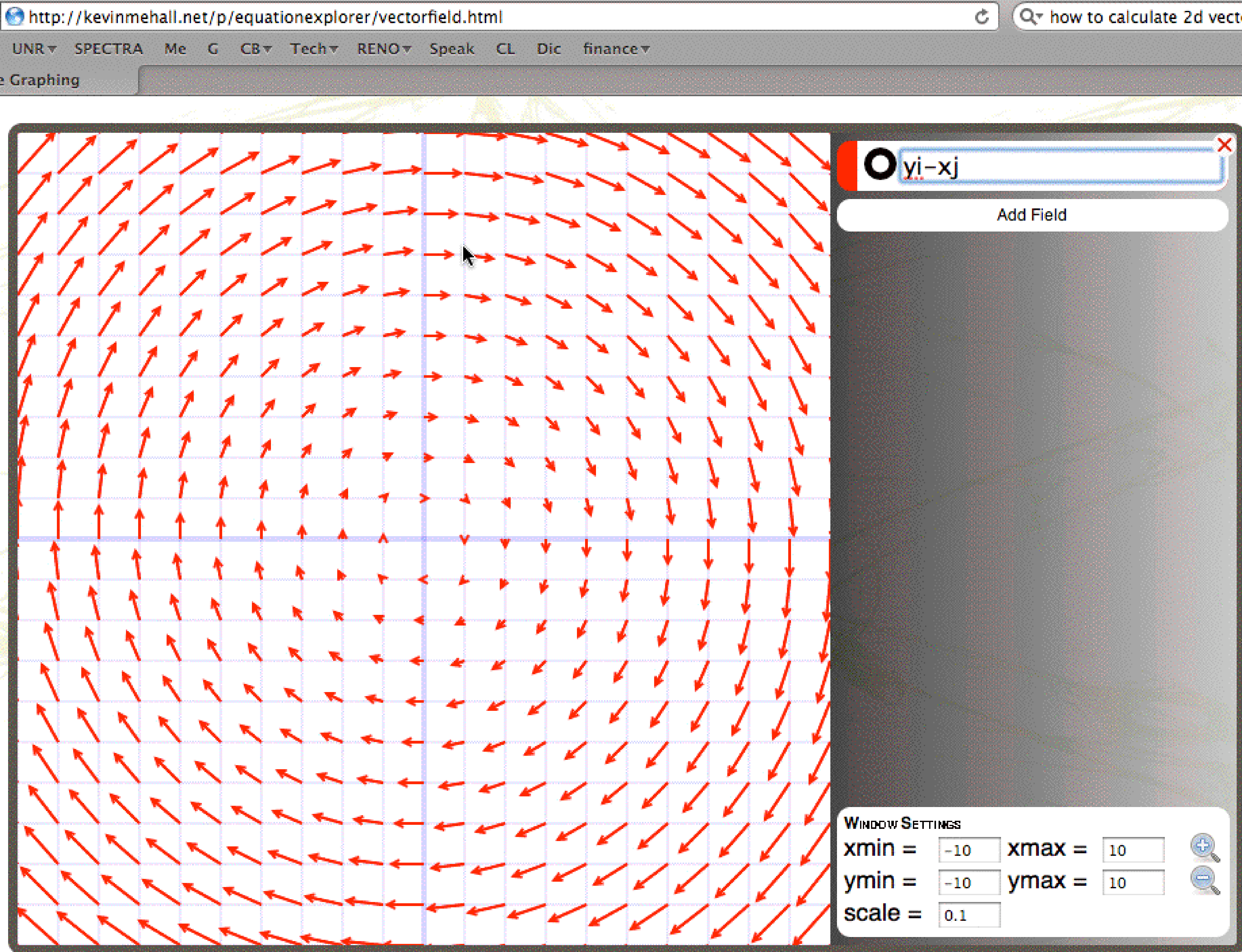
scale = 0.1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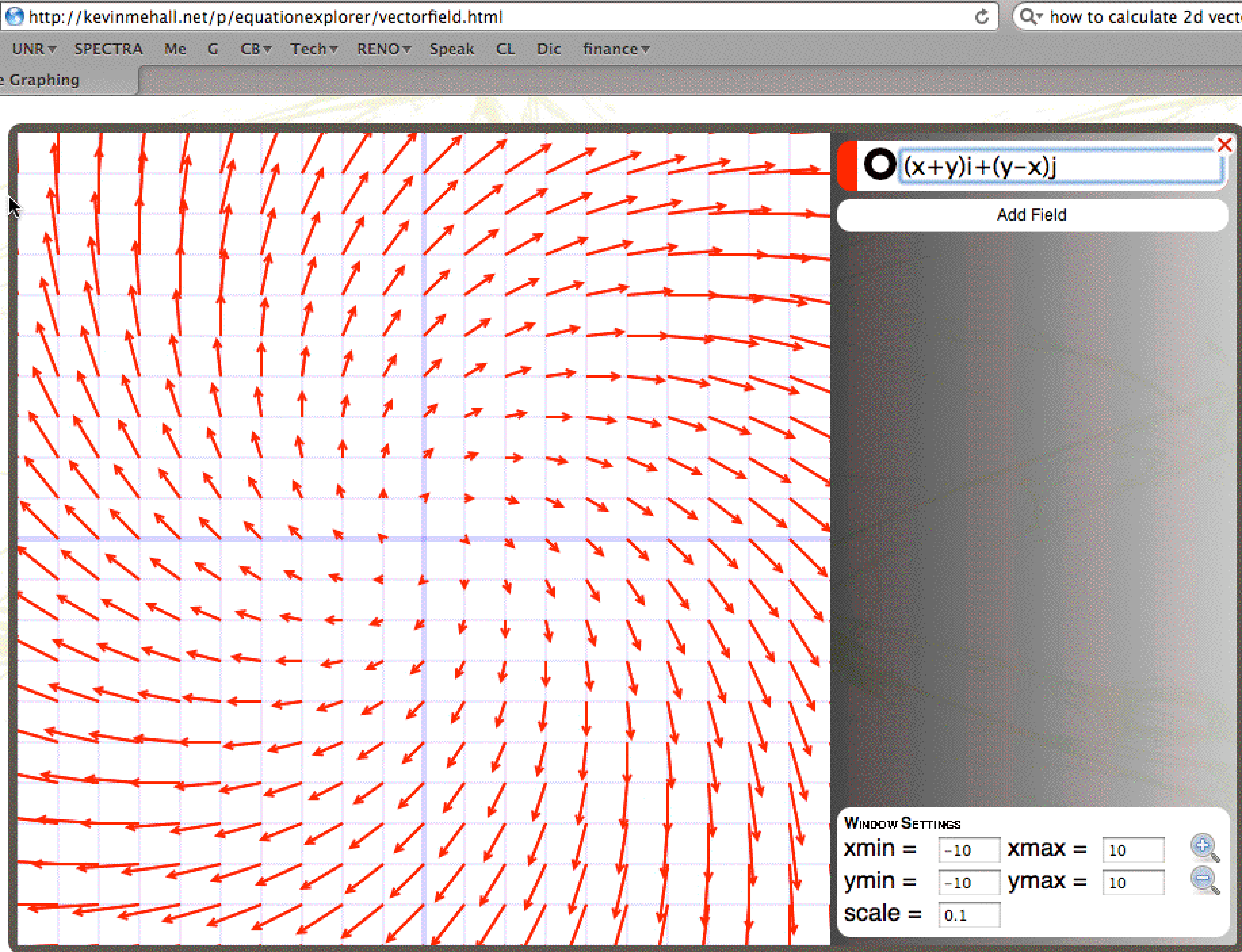

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