Chapter 3 Properties of fluids

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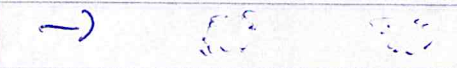
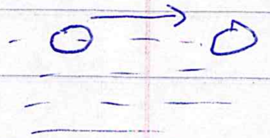
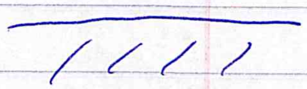




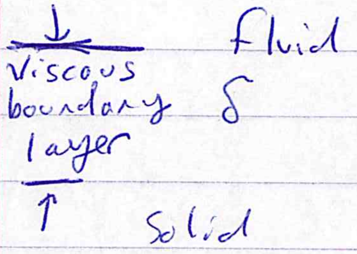
Fluid

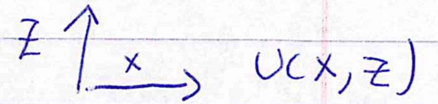
Solid



Applying a stress causes the fluid parcel to relocate,

applying a stress the solid moves but will restore to original position when released





oscillating boundary, f = frequency

Boundary Condition: No slip of fluid solid-boundary at interface.

$$\delta = \left(\frac{2\eta}{\rho\omega}\right)^{1/2}$$

$\omega = 2\pi f$

One definition of viscosity η

For air

$$\eta = 1.98 \times 10^{-5} \, \frac{kg}{m\,s} \left[Pa\,s \right] \quad Pa = \frac{kg}{m\,s^2} = \frac{\text{Force}}{\text{Area}}$$

Newton

Roughly $\eta(T) \propto T^{1/2}$

└── Form of a transport

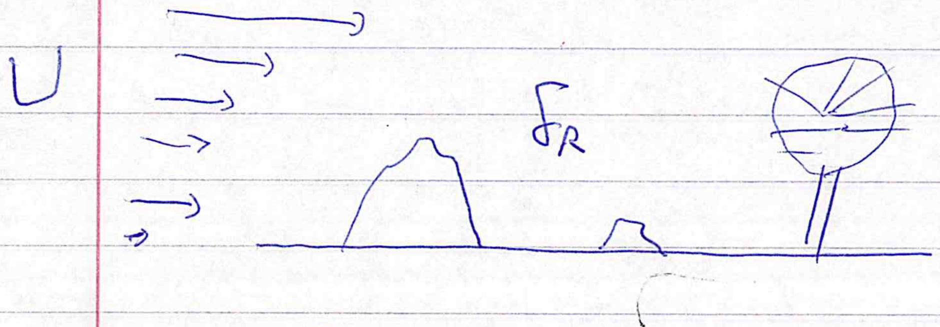
Example: Air @ 300K $\rho = 1 \, kg/m^3$

Length over which a fluid (air) supports velocity shear as a function of frequency

f (Hz)	δ (meters)	
0.01	2.5×10^{-2}	2.5 cm
0.1	7.9×10^{-3}	8 mm
1	2.5×10^{-3}	2.5 mm
10	0.8 mm	
100	0.25 mm	
1000	79 μm	

Sound frequencies in atmosphere, Smooth boundaries interact only with over about 1 hair width away. Many Surfaces are not smooth on an 80 μm length scale.

Analogy



Flow Profile in boundary layer

δ_R = Surface roughness scale
~ 1 to 100 meters land
~ 0.1 to 10 meters ocean.

Flow dragged, blocked, and redirected by Surface roughness elements.

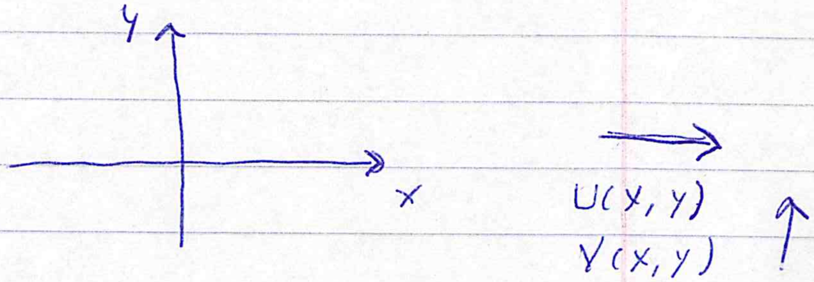
Not Covered

Midlatitude Atmos dynamics Covered in Ch2 of lynch with my Problem

§ 1.4 Basic Kinematics of fluids from Mid latitude atmospheric dynamics by Jonathan Martin, [very good general description and introduction of Vorticity and divergence]

Take a 2-D fluid Problem:

Need fluid equations to see how they come about



Consider slight perturbations of the fluids:

(1) $$U(x,y) = U_0 + \left.\frac{\partial U}{\partial x}\right)_0 x + \left.\frac{\partial U}{\partial y}\right)_0 y + H.O.T.$$

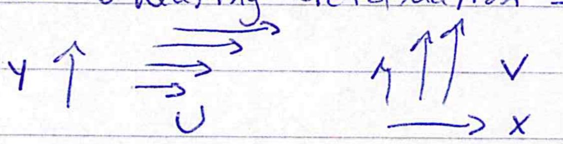
(2) $$V(x,y) = V_0 + \left.\frac{\partial V}{\partial x}\right)_0 x + \left.\frac{\partial V}{\partial y}\right)_0 y + H.O.T.$$

Define: $D \equiv$ divergence $= \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}$

$F_1 \equiv$ Stretching deformation $= \frac{\partial U}{\partial x} - \frac{\partial V}{\partial y}$

[Positive definite if U gets larger along x, V smaller along y]

$F_2 =$ Shearing deformation $= \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}$



$\zeta \equiv$ Vorticity $\equiv \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}$ Extremely important

Not Covered

Cont

Using definitions and algebra,

Perturbation of U and V

$$\tilde{U} \equiv U(x,y) - U_0 = \frac{1}{2}\left(Dx + F_1 x - \zeta y + F_2 y\right)$$

$$\tilde{V} \equiv V(x,y) - V_0 = \frac{1}{2}\left(\zeta x + F_2 x + Dy - F_1 y\right)$$

Now we can study each in isolation:

§ 1.4.1 <u>Pure Vorticity</u>; Set $\zeta = 1$

$F_1, F_2, D = 0$

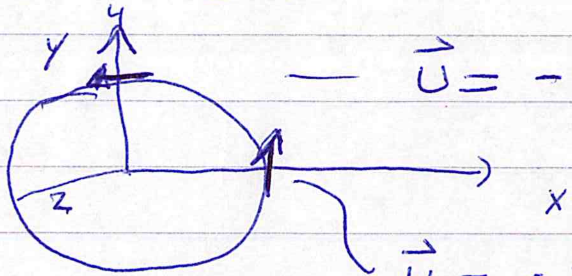
$$\tilde{U} = -\tfrac{1}{2} y \qquad \vec{U} = \tilde{U}\hat{x} + \tilde{V}\hat{y}$$

$$\hat{V} = \tfrac{1}{2} x$$

To graph, choose a contour $\sqrt{\tilde{U}^2 + \tilde{V}^2} = 1$

(wind speed = 1)

$$\frac{x^2}{4} + \frac{y^2}{4} = 1 \qquad x^2 + y^2 = 2^2$$



— $\vec{U} = -1\hat{x} + 0\hat{y}$

$\vec{U} = 0\hat{x} + 1\hat{y}$

Positive Vorticity has flow in CCW direction

Not Covered

Cont

§ 1.4.2 Pure divergence

Let $D = 1$, ζ , F_1 , $F_2 = 0$

$$\tilde{U} = \frac{1}{2} x = \frac{dx}{dt}$$

$$\tilde{V} = \frac{1}{2} y = \frac{dy}{dt}$$

} Position Change with time

Fluid flow is a function of x and y.
[from x_0, y_0, t_0 to x, y, t]

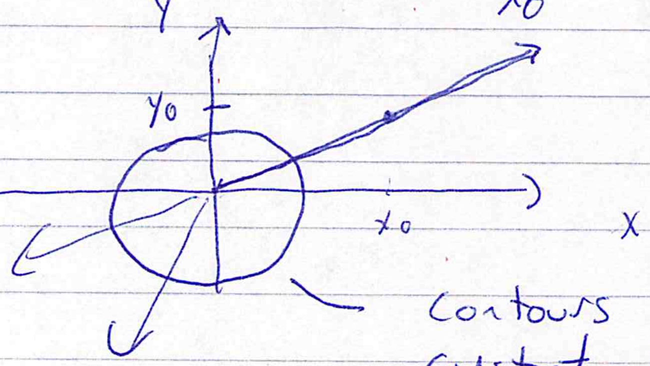
Calculate Pathlines of fluid [trajectory analysis]

$$\frac{dx}{dt} = \frac{x}{2} \qquad \int_{x_0}^{x} \frac{dx'}{x'} = \frac{1}{2}\int_{t_0}^{t} dt'$$

$$\ln \frac{x}{x_0} = \frac{t - t_0}{2} = \ln \frac{y}{y_0}$$

Pathlines are $y = \frac{y_0}{x_0} x$

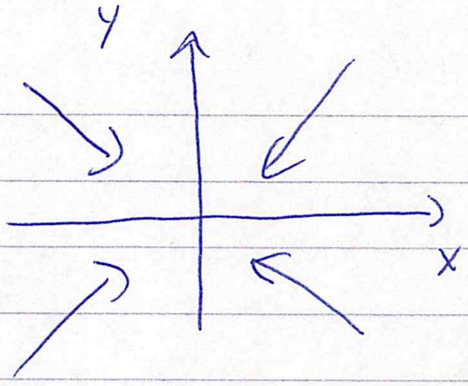
Fluid diverges from the Center



contours of constant $|\vec{U}|$ are circles

$\boxed{\text{Not Covered}}$ Cont

Pg 3.6



For $D = -1$ get fluid

Convergence towards Center

[Need to have conservation equations, mass, energy, etc, to show how we can physically have D and ζ in a fluid].

Pure Stretching § 1.4.3

$F_1 = 1 \qquad D, \zeta, F_2 = 0$

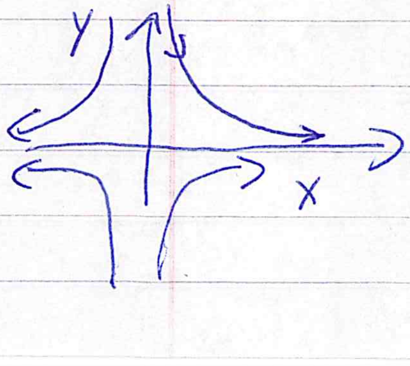
$$\tilde{U} = \tfrac{1}{2} x = dx/dt$$
$$\tilde{V} = -\tfrac{1}{2} y = dy/dt$$

Path lines

$$\frac{dx}{x} = \frac{dt}{2} \qquad x = x_0 e^{+(t-t_0)/2}$$

$$\frac{dy}{y} = \frac{-dt}{2} \qquad y = y_0 e^{-(t-t_0)/2}$$

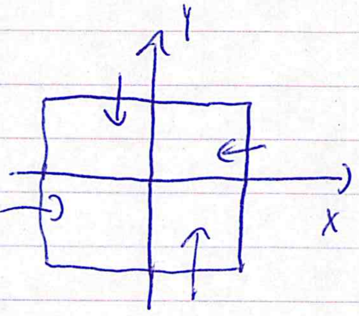
$$\Rightarrow \quad y = \frac{x_0 y_0}{x}$$



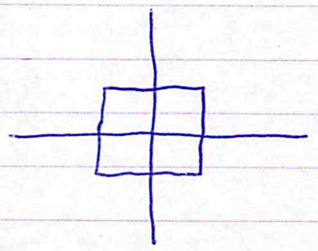
Not Covered

Cont

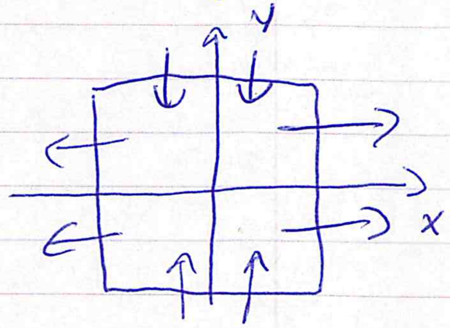
Divergence (convergence) Versus Stretching (contracting)



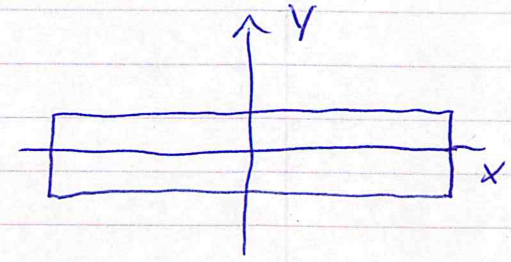
Convergence



(Box gets smaller)



Stretching



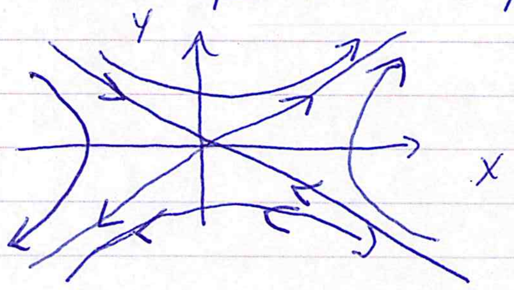
[Box turns into rectangle]

§ 1.4.4 Pure shearing deformation

$F_2 = 1$

$\tilde{U} = 1/2\, y \qquad \tilde{V} = 1/2\, x$

Pathlines are hyperboli $\quad y^2 - x^2 = y_0^2 - x_0^2$

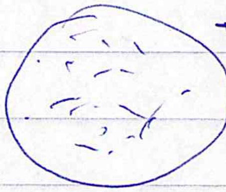


Different Pathlines for different choices of x_0 y_0

§ 3.3.1 Dry atmosphere - adiabatic lapse rate.

Dry Air : 78% N_2 21% O_2 1% other - CO_2, Ar, etc

Ideal Gas LAW :



P = Pressure

V = volume

N = # molecules

T = Temperature

Point like billiard balls bouncing around

Fundamental form

$$PV = NkT$$

$k = 1.38 \times 10^{-23}$ J / molecule °K

Daltons Law partial pressure

$$\Rightarrow PV = (N_{N_2} + N_{O_2} + N_{CO_2} + N_{AR} + \dots) kT$$

$$P = \underbrace{\frac{N}{V} \frac{\mu \text{ Kg}}{\text{mole}} \frac{\text{mole}}{N_a \text{ molecules}}}_{\rho} \underset{R_{dia}}{\frac{N_a k}{\mu}} \; T$$

ρ = density

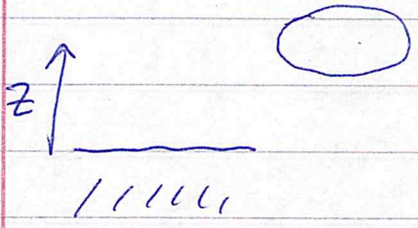
N_a = Avogadros #

$\mu = 29 \frac{\text{Kg}}{\text{mole}}$ molecular weight dry air

$$P = \rho R_d T$$

$$R_d = 287 \frac{J}{\text{Kg } °K}$$

$$\Gamma_0 = \underbrace{-dT/dZ = g/c_p}_{\text{Adiabatic Lapse Rate}}$$ Pg 3.9



Air Parcel created here

$P, V, T, N, M = \frac{N\mu}{N_a}$

(fixed

E

Enthalpy $\rightarrow$ $H(Z) = \frac{5}{2} kTN + MgZ + PV$

$\frac{5}{2}kTN$: internal Energy [K.E.]

$\frac{1}{2}m(u^2+v^2+w^2) + \frac{1}{2}I(\omega_x^2+\omega_y^2)$

5 Energy Pathways in Equilibrium with each other at temperature T

MgZ: P.E.

PV: ~~K~~work needed to "blow up" the parcel to Volume V against the pressure P of the surroundings.

$H \equiv E + PV$

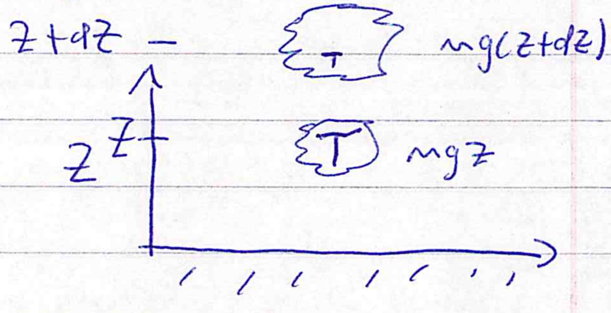
$\therefore$

$dH = dE + PdV + VdP$

$dE = \text{đ}Q - PdV$

$dH = \text{đ}Q + VdP$

Processes at constant P involve only heat transfer đQ



Parcel Neutrally Stable if

$$H(Z+dZ) = H(Z)$$

$$\cancel{\frac{5}{2}NkT} + \frac{5}{2}NkdT + \cancel{mgZ} + mgdZ + \overset{PV + PdV + VdP}{(P+dP)(V+dV)}$$

$$= \cancel{\frac{5}{2}NkT} + \cancel{mgZ} + \cancel{PV}$$

$$\frac{5}{2} N k\, dT + \frac{\mu N}{N_a} g\, dz + \underbrace{P dV + V dP}_{\text{I.G.L.}} = 0$$

I.G.L.

$$PV = NkT$$

$$P\,dV + V\,dP = Nk\,dT$$

Multiply by $\frac{N_a}{\mu N}$

$$\underbrace{\frac{7}{2} \frac{N_a k}{\mu}}_{C_p = \frac{7}{2} R_d} dT + g\,dz = 0$$

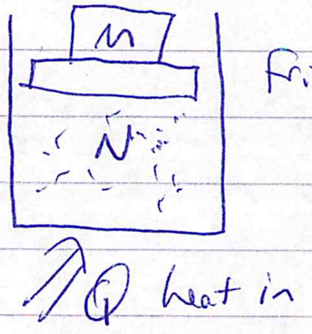
$C_p = \frac{7}{2} R_d$ Specific heat at Constant Pressure

$$\boxed{\frac{dT}{dz} = -\frac{g}{C_p} = -\Gamma_D}$$

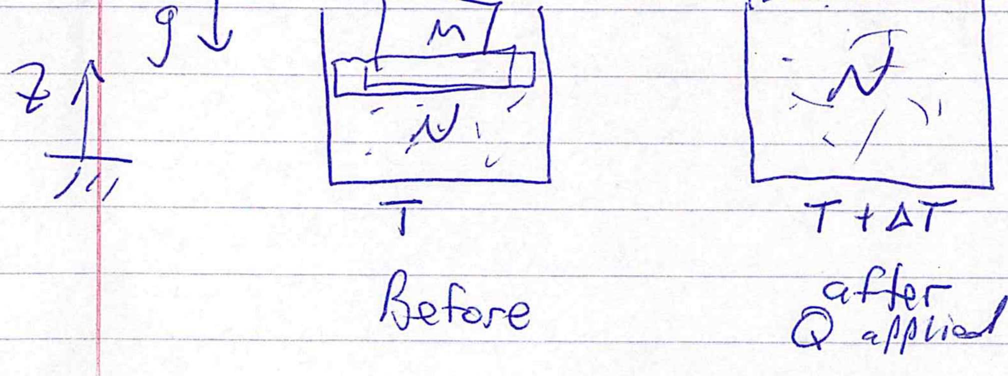
$$C_p = 1004 \frac{J}{kg\ °K}$$

Aside

Definition of C_p



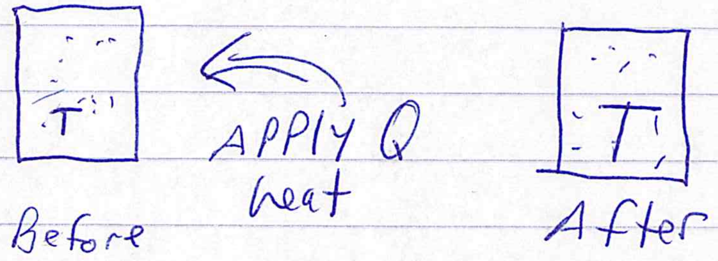
$$\frac{N \mu}{N_a} C_p \Delta T = Q$$



Some Q needed to raise mass M up, some goes to T increase.

By Contrast

V = constant



T greater than constant Pressure Process

$$\frac{N_g}{N_e} C_v \Delta T = Q$$

$$C_v = \frac{5}{2} R_d$$

$$C_p - C_v = R_d$$

$\boxed{PV^\gamma = \text{constant for a gas, adiabatic process}}$ $\gamma = C_p/C_v = 7/5 = 1.4$ for dry air

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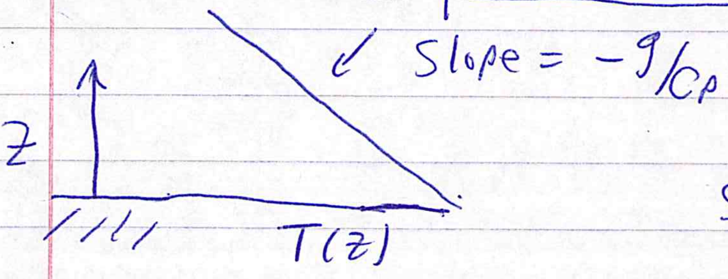
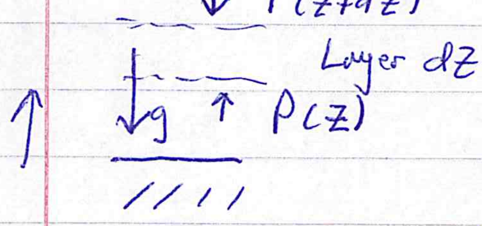


Potential Temperature

Surface, $Z=0$ $P = 1013.25$ mb $\equiv P_0$

$$\frac{dT}{dZ} = -\frac{g}{C_p} \Rightarrow T(Z) = T_0 - \frac{g}{C_p} Z$$



Force Balance

Hydrostatic Equilibrium

$$(\underset{\text{Layer Mass}}{\rho} A dZ) g + P(Z+dZ)A = P(Z)A$$

$$P(z+dz) \cong P(z) + \left(\frac{\partial P}{\partial z}\right) dZ$$

$$= P(z) + dP$$

So

$$\boxed{\rho g \, dZ = -dP}$$

$dP = -\rho g \, dZ$

Hydrostatic Equation

$$\boxed{\frac{dP}{\rho} = -g \, dZ}$$

So for

$$\frac{dT}{dZ} = \frac{-g}{C_p}$$

$$C_p \, dT = -g \, dZ = \frac{dP}{\rho}$$

For the ideal gas

$$P = \rho R_d T \qquad \rho = \cancel{\frac{P}{R_d T}} \frac{P}{R_d T}$$

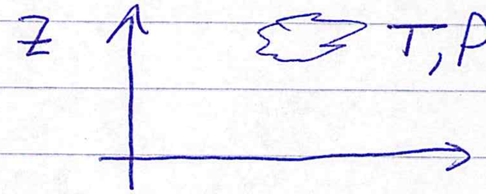
Then $$\int_{P_0}^{P} \frac{dP}{P} = \frac{C_p}{R_D} \int_{\Theta}^{T} \frac{dT}{T}$$

← definition of Potential T, T where $P = P_0$ = Seal Level

$$\ln \frac{P}{P_0} = \frac{C_p}{R_D} \ln \frac{T}{\Theta}$$

$$\Theta = T \left(\frac{P_0}{P}\right)^{R_D/C_p}$$

Summarizing :



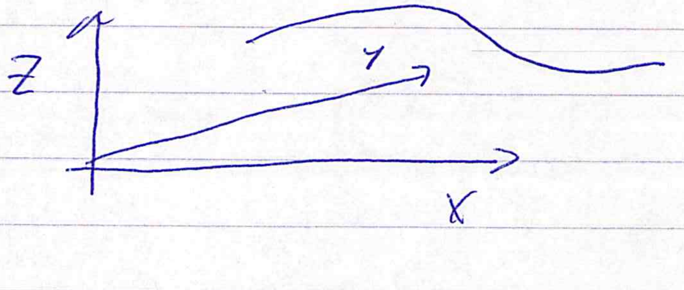
air parcel at T(z), P(z)

If lowered adiabatically to $z=0$ sea level where $P(z=0) = P_o$, then

$$T(z=0) \equiv \Theta = T\left(\frac{P_o}{P}\right)^{R_d/C_p}$$

where $\frac{R_d}{C_p} = \frac{2}{7}$

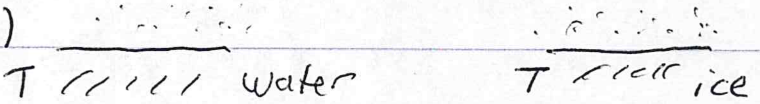
Important because atmosphere is close to adiabatic at times;



Θ = constant on this line

Line of contour of constant Θ is known as isentrope ⇒ no heat needed to be added to have air parcels have the same entropy ⇒ winds tend to follow isentropes. Entropy quantifies heat added to systems.

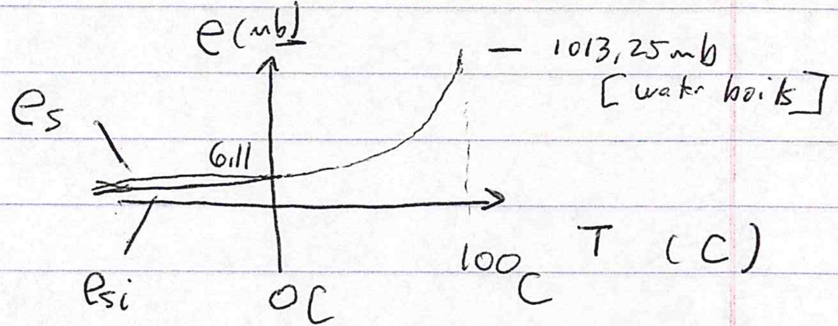
§ 3.3.2 Moist atmosphere

$e_s(T)$ 

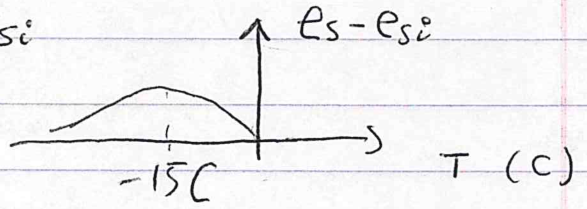
$e_{si}(T)$

(Saturation vapor pressure over water

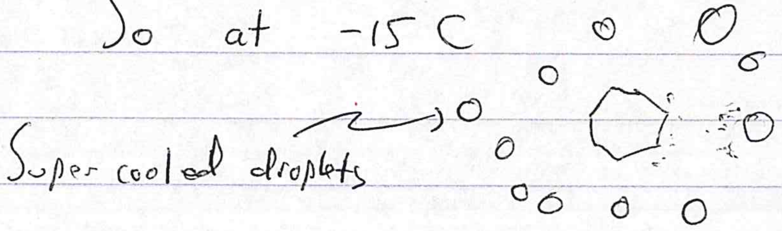
(Saturation vapor Pressure over ice



Note: $e_s \geq e_{si}$



So at -15 C



ice crystals grow at the expense of water droplets.

[Known as the Wegener - Bergeron Findeisen Process]

If # of ice nuclei << # Cloud droplets ⇒ ice will grow large enough to fall out.

For dry air: $P = \rho R_d T$

For moist air we must add in the number of water vapor molecules per unit volume.

$$R_d = R^* \left[\frac{J}{\text{mole K}}\right] \frac{1000 \text{ moles}}{29 \text{ Kg}} \qquad \left[\text{Gas constant for dry air}\right]$$

$$R_v = R^* * \frac{1000}{18} \frac{J}{\text{Kg °K}} \qquad \left[\text{Gas constant for water vapor}\right]$$

Moist air:

(1) $$P = (\rho_d R_d + \rho_v R_v) T$$

where

$$\rho = \underbrace{\rho_d}_{\text{dry air}} + \underbrace{\rho_v}_{\text{water vapor}}$$

Define:

$$\boxed{\frac{1}{q} = \frac{1}{w} + 1}$$

can show

$$\underbrace{q}_{\text{specific humidity}} \equiv \rho_v/\rho \qquad 1 - q = \rho_d/\rho$$

Note: $W \equiv$ water vapor mixing ratio $= \frac{\rho_v}{\rho_d}$ so
[g/Kg] typical
$w > q$ by a few percent at most.
$W \approx q$

Algebraically,

$$\frac{R_v}{R_d} = \frac{\mu_d}{\mu_v}$$

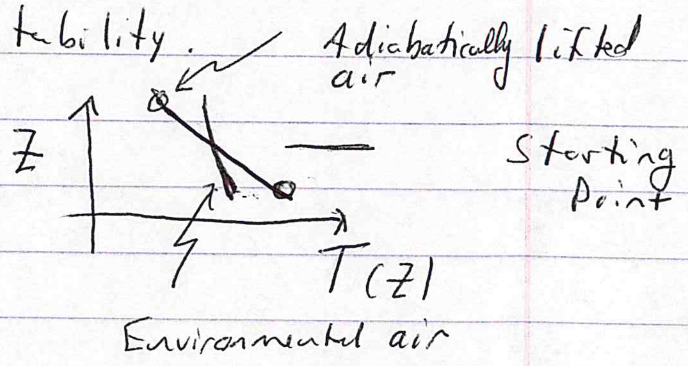
$$P = \rho R_d \underbrace{\left[(1-q) + q\, R_v/R_d\right] T}_{\text{virtual temperature } T_v \text{ definition}}$$

$T_v \geq T$. ρ moist air $\leq \rho_{dry}$ at same P & T

§ 3.4 Static Stability.

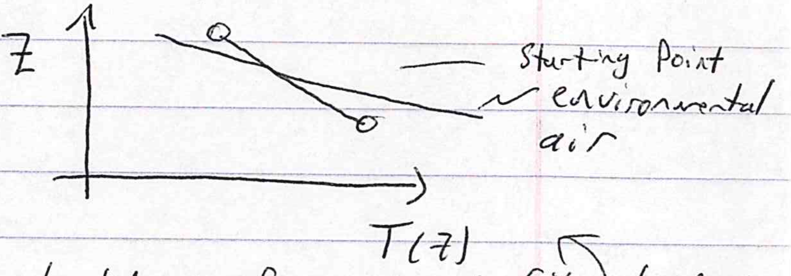
Key message:

Stable



Lifting [lowering] a parcel from starting point makes it cooler [warmer] than the environmental air at the same level, so the air would return, and maybe oscillate, about the starting point.

Unstable



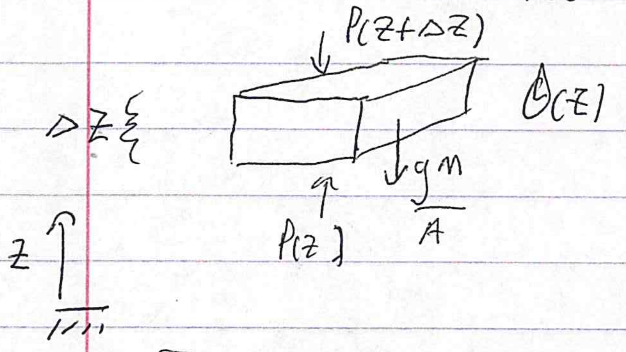
Near the midday solar heated surface this profile happens the lifted air  is now warmer than surroundings so is less dense, so it rises up!.

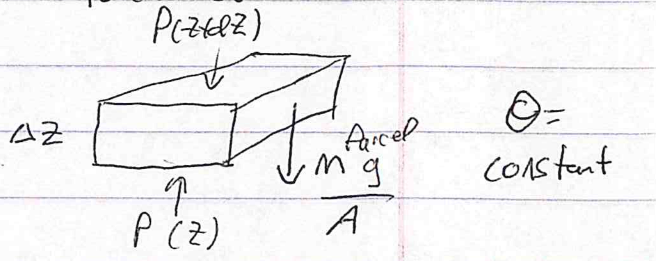
If the air is at saturation, lifting occurs along a moist adiabat.

[will do a section of skew T log P to illustrate phenomena at end of chapter]

Archimedes Principle:

Where does the boyant force come from?





test Parcel

Environment:

$$\frac{\overrightarrow{\text{Force}}}{\text{Area}} = \hat{z}\left[\overbrace{P(z)-P(z+dz)} - \overbrace{\int_{z}^{z+\Delta z}\rho(z')g\,dz'}\right]$$

$$= 0 \quad \text{For hydrostatic equilibrium}$$

Test Parcel: [We lift somehow]

$$\frac{\overrightarrow{\text{Force}}}{\text{Area}} = \hat{z}\left[\underbrace{P(z)-P(z+dz)} - \int_{z}^{z+\Delta z}\rho^{\text{Parcel}}(z')g\,dz'\right]$$

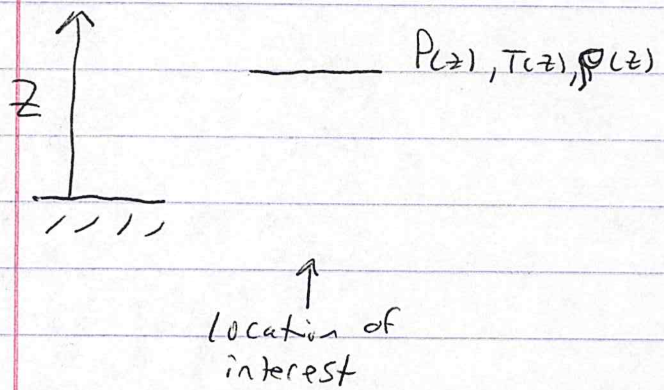
$$\approx \hat{z}\left(\rho(z+\Delta z) - \rho^{\text{Parcel}}(z+\Delta z)\right)g\,\Delta z$$

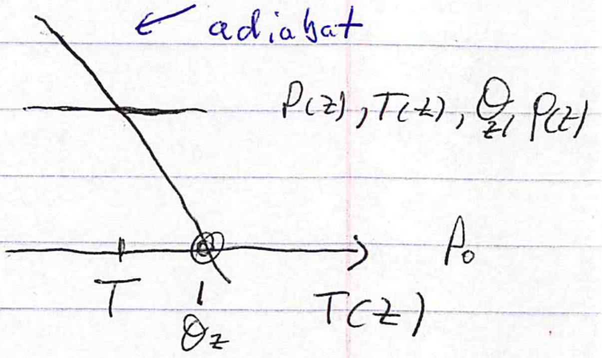
if Parcel is more dense [colder] $\vec{F} < 0$
so Parcel restores to original Position.

[Another look at Parcel theory]

Brunt-Väisälä frequency:

Very useful to understand gravity waves in the atmosphere – in the lee of mountains.





test Parcel at Z.
Characterized by
$\theta_z \Rightarrow$

$$T_{(z)} = \theta_z \left(\frac{P(z)}{P_0}\right)^{R_d/c_p}$$

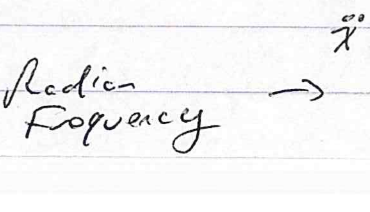
A parcel displaced by dZ has a boyant force acting on it. Acceleration is

$$\frac{1}{\theta}\frac{d\theta}{dZ} = \frac{d\ln\theta}{dZ}$$

$$\boxed{\ddot{Z} = -\underbrace{\frac{g}{\theta(z)}\frac{d\theta(z)}{dZ}}_{N^2} dZ = -N^2 dZ}$$

Could do first

Analogous Problem



$F = m\ddot{x} = -k\,dx$

$\ddot{x} = -\frac{k}{m} x$

$x = Ae^{i\omega t}$

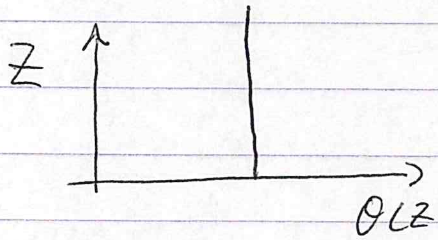
$\ddot{x} = -\omega^2 x$

$x = A\cos\omega t$

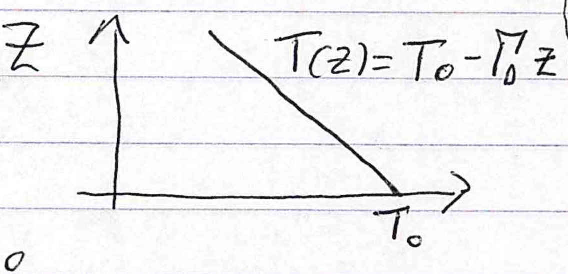
$\ddot{x} = -\omega^2 x$

Radian Frequency $\rightarrow$ $\omega = \sqrt{\frac{k}{m}}$

Case 1. $\theta(z) = \text{constant}$ $\quad \frac{d\theta}{dz} = 0$



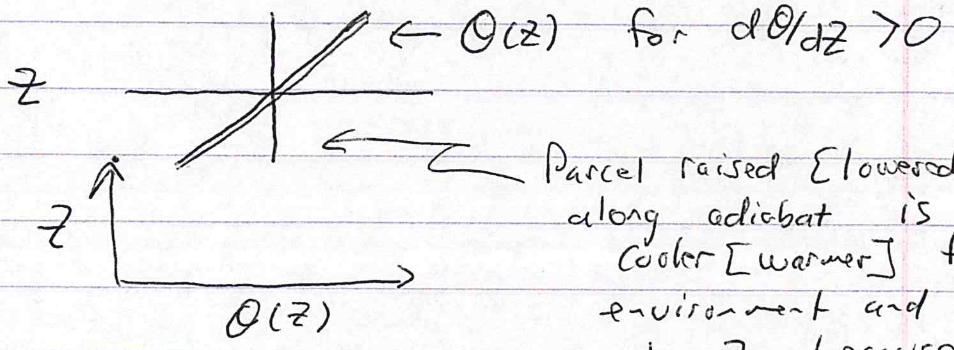
well mixed parcel that has T(z) follow the dry adiabat



$$\Gamma_D = -\partial T/\partial z\Big)_{adiabat} = g/C_p$$

$\ddot{z} = 0$ in this case.

Case 2: $\frac{d\theta}{dz} > 0$ $\quad N^2 = \frac{g}{\theta(z)}\frac{d\theta(z)}{dz} > 0$

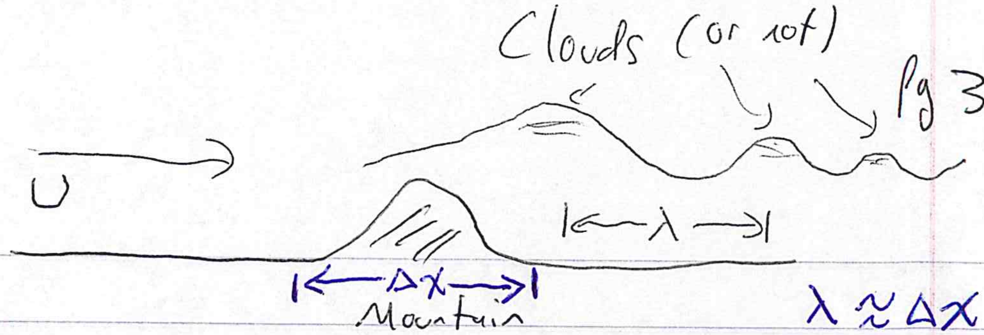


Parcel raised [lowered] along adiabat is cooler [warmer] than environment and returns to z because its density is higher [lower]

$\ddot{z} = -N^2 z$ $\quad z = A\cos\omega t$ $\quad \omega = N$

More stable parcels have larger $\frac{d\theta}{dz}$, larger N, so oscillate faster, $\dot{z} = -\omega A\cos\omega t = -NA\cos Nt$ than less stable parcels.

Pg 3,19

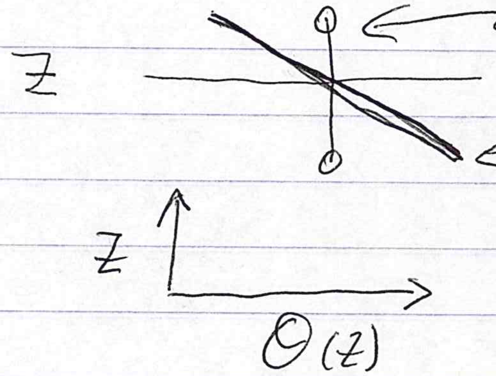


$\lambda \approx \Delta x$

Mountain waves have standing wave patterns where $\lambda \approx 2\pi \frac{U}{N}$. Large N create small λ.

Often visible in satellite imagery.

Case 3: $\frac{d\theta}{dz} < 0$ $\qquad N^2 = \frac{g}{\theta}\frac{d\theta}{dz} < 0$



← $\theta(z)$ = constant - test parcel

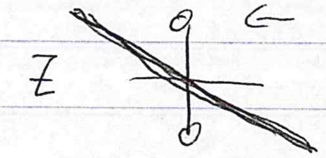
← environmental $\theta(z)$

$\ddot{z} = |N|^2 z$

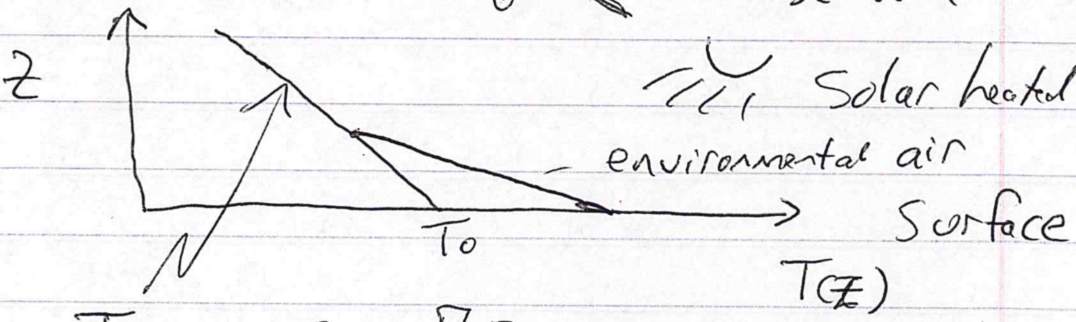
unstable growth $z = Ae^{|N|t}$

$\ddot{z} = |N|^2 z$

Now a parcel raised above z will be unstable ⇒ will continue to rise



← warmer than environment - less dense, will rise more



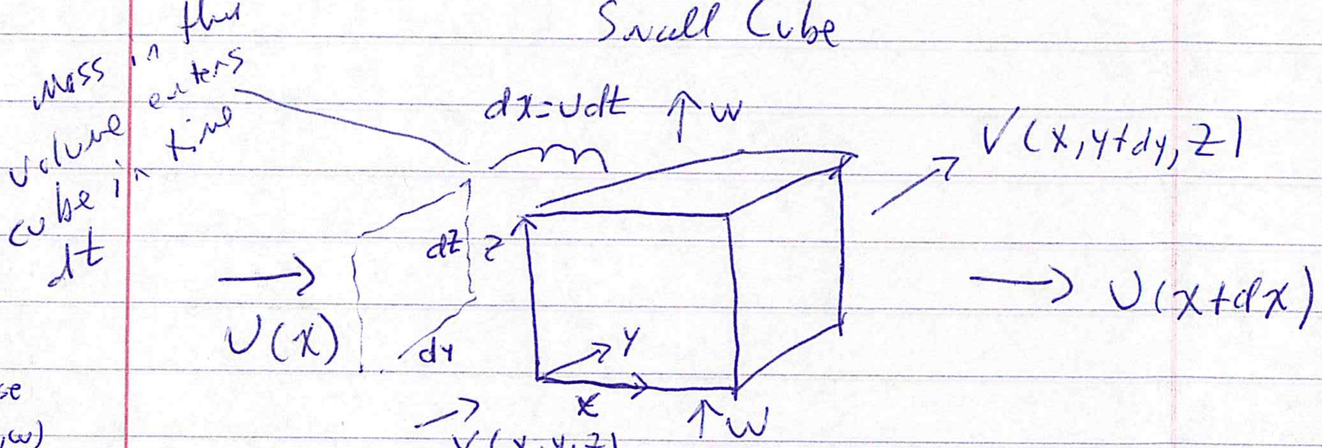
$T(z) = T_0 - \Gamma_D z$

$\theta(z) = T_0$ = constant

Continuity Equation for Mass
Conservation of Mass

Pg 3.20

Small Cube



Choose (U,V,W) directions for illustration sign takes care of it.

Time rate of Changed of mass in volume

$$\dot{M} = \frac{\partial}{\partial t} \rho(x,y,z)\,dx\,dy\,dz =$$

rate of mass in − rate of mass out

$\frac{kg}{m^3}\frac{m}{s}m^2$

$$= \rho U(x)\,dy\,dz - \overset{\rho(x+dx,y,z)}{\rho} U(x+dx,y,z)\,dy\,dz + \qquad (1)$$
$$\rho V\,dx\,dz - \rho V(y+dy)\,dx\,dz +$$
$$\rho W\,dx\,dy - \rho W(z+dz)\,dx\,dy$$

$$\rho U(x+dx,y,z) \approx \rho U(x,y,z) + \frac{\partial \rho U}{\partial x}(x,y,z)\,dx$$

Same for V and W.

(1) becomes $- \partial/\partial x\, \rho U(x,y,z)\,dx\,dy\,dz$
$- \partial/\partial y\, \rho V\,dx\,dy\,dz - \partial/\partial z\, \rho W\,dx\,dy\,dz$

$\Rightarrow$

$$\frac{\partial \rho(x,y,z,t)}{\partial t} + \frac{\partial}{\partial x}\rho U + \frac{\partial}{\partial y}\rho V + \frac{\partial}{\partial z}\rho W = 0$$

Chain Rule:

$$\boxed{\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\frac{\partial u}{\partial x} + \rho\frac{\partial v}{\partial y} + \rho\frac{\partial w}{\partial z} = 0}$$

Vector Notation:

$$\boxed{\underset{\text{rate of change of } \rho \text{ at } x,y,z}{\frac{\partial \rho}{\partial t}} + \underset{\text{Gradient of } \rho}{\vec{U}\cdot\vec{\nabla}\rho} + \underset{\text{Divergence of } \vec{U}}{\rho\,\vec{\nabla}\cdot\vec{U}} = 0}$$

Example

Using

$$\cancel{\partial D}\ \frac{D\rho}{Dt} = \left(\frac{\partial}{\partial t} + \vec{U}\cdot\vec{\nabla}\right)\rho = -\rho\,\vec{\nabla}\cdot\vec{U}$$

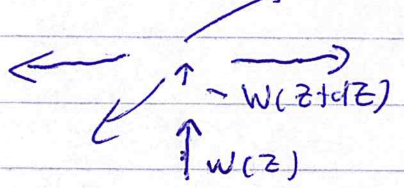
└ Rate of change of density along air parcel

Change of density in a moving $\vec{V}$ parcel is due to divergence. $\vec{\nabla}\cdot\vec{U} > 0$ divergence, flow out, $\Rightarrow$ $\frac{D\rho}{Dt} < 0$

Example

If ρ = constant, $\partial\rho/\partial t = 0$ $\nabla\rho = 0$,

$$\vec{\nabla}\cdot\vec{U} = 0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

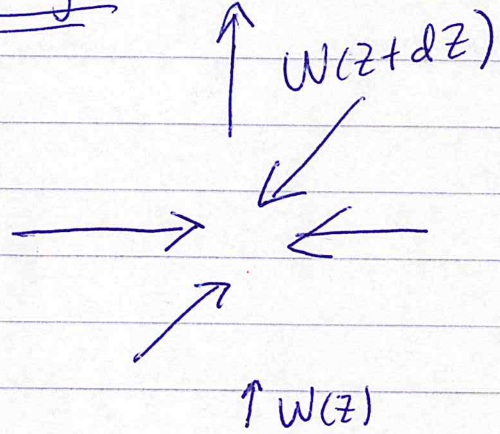


Divergence $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} > 0$

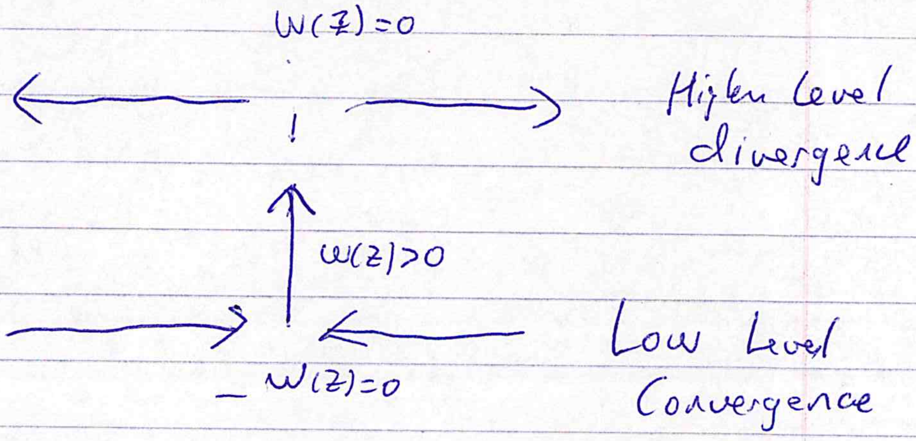
$\Rightarrow \frac{\partial w}{\partial z} < 0$

Similarly if $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} < 0$, $\frac{\partial W}{\partial Z} > 0$

Convergence



Flow must increase in vertical to allow for horizontal convergence



Known as Dines compensation.

Need to combine the continuity Eq with the equation of motion and equation of state to see how the atmosphere works from an analyis perspective,