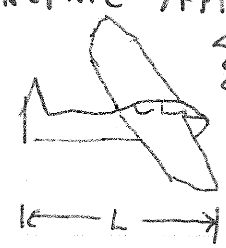
Chapter 5 Scale Analysis:

weather balloons

Purpose: A) Characterize nature of flow at various scales of time and space in the atmosphere.
B) Learn how to make various approximations.
[Geostrophic Approximation]

Example



$\Leftarrow U$ when making airplanes its a good idea to build a toy prototype and to test it in a wind tunnel

inertia ⇒ stay in motion

How: Scale size and flow speed U to be similar to what it will be for the real thing.

Reynolds #: $Re \sim \dfrac{\text{inertial acceleration}}{\text{viscous acceleration}}$

Re small, viscous acceleration dominates ⇒ laminar flow
Re ≳ 2000 – turbulent flow

$\partial U/\partial t$

$$Re \sim \left| \frac{\partial U/\partial t}{\nu\left(\partial^2 U/\partial x^2 + \partial^2 U/\partial y^2 + \partial^2 U/\partial z^2\right)} \right|$$

For scaling: Note units [very simple idea]

$u \sim U \sim L/T$ $\qquad$ U = Velocity Scale
L = Characteristic Length
T = Characteristic Time

$\nu = \dfrac{\eta}{\rho} = \dfrac{\text{Viscosity}}{\text{density}}$

$$Re \sim \frac{U/T}{\nu\, U/L^2} \qquad T \sim L/U \sim LU^{-1}$$

$$Re \sim \frac{U T^{-1}}{\nu U L^{-2}} \sim \frac{U L^{-1} U}{\nu U L^{-2}} \sim \frac{UL}{\nu}$$

$$Re \sim \frac{\rho U L}{\eta}$$

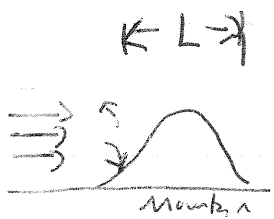
$\Rightarrow$ If scale model is 1/10 as big as real thing, have to make $\rho U/\eta$ 10 times larger for same Re, same aerodynamics.

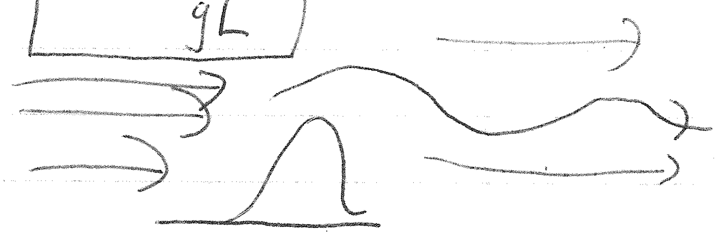
Similarly:

Froude #: $$Fr \sim \frac{\text{inertial}}{\text{gravity}} \simeq \frac{\partial U/\partial t}{g}$$

$1/t \sim U/L$

$$\boxed{Fr \sim \frac{U^2}{gL}}$$





$Fr \ll 1$

Flow tends to spread out, not over mountain.

$Fr \gg 1$

Flow over mountain $\Rightarrow$ gravity waves if a stable layer exists.

At what scales is the Coriolis force important?

$$Ro \equiv \text{Rossby \#} \sim \frac{\text{Inertia}}{\text{Coriolis}}$$

U component analysis Still dominant

$$\sim \frac{\partial U/\partial t}{2\Omega V\sin\phi - 2\Omega w\cos\phi}$$

$$\sim \frac{L/T \cdot T^{-1}}{2\Omega L/T}$$

$T \sim L/U$

$$\sim \frac{1}{2\Omega T} \sim \boxed{\frac{U}{2\Omega L} = Ro}$$

When $Ro < 1$, Coriolis > Inertia
Small Rossby #s ⇒ Rossby Waves ⇒ "Coriolis Force"

When $Ro > 1$, Inertia dominates.

Here Ω = rotation rate of Earth

$$= \frac{2\pi \text{ radians}}{\text{day}} = \frac{2\pi}{86{,}400 \text{ secs}} = \frac{7.27\times10^{-5}}{s}$$

1 day = 86 400 seconds

so

$$2\Omega = 1.454 \times 10^{-4} \text{ /second}$$
$$\sim 10^{-4}\, s^{-1}$$

<u>Examples</u>: $R_o \sim U/2\Omega L$

$U = L/T$

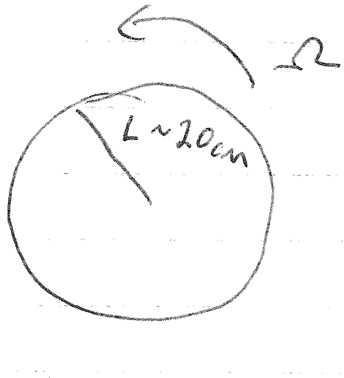
Not at all getting near 50

<u>Item</u>	<u>L</u>	<u>T</u>	<u>U</u>	<u>R_o</u>	<u>Coriolis</u>?
Bath Tub	1 m	10 s	0.1 m/s	10^3	X
Tornado	1000 m / 100m	10 min	~ 2 m/s	10	X
{Surface low System}	1000 km	2 days	5 m/s	0.05	✓
{Planetary waves}	10000 km	week to month ~10^6 sec	$10\frac{m}{s}$	0.01	✓

<u>Summary</u>: Rossby waves are important on the synoptic and global scale ⇒ coriolis force matters! on these scales ⇒ Small scale phenomena like draining the bath tub ⇒ irrelevant w.r.t. coriolis.

$\Omega \simeq 2$ rpm
$= \frac{1}{30}\,s^{-1}$

$U \sim \Omega L$



Rotating tank ⇒ in lab

$R_o \sim 0.5$

Can expect to see effects of coriolis force on flow in this system where we rotate with a motor!

Pg 5.5

nonlinear terms are the advection terms, $u\partial u/\partial x$ and $v\partial u/\partial y$ for example. In nonlinear systems, different solutions cannot be superposed (added together) to form new solutions. This makes solving the equations much more difficult than linear systems. Hence, our approach in this book will be to simplify the equations in order to solve them.

Since much of the significant weather in middle latitudes is associated with cyclones and, to a lesser extent, anticyclones, we will derive a simpler form of the equation specifically for these systems. Typical values for the appropriate scales are shown in Table 5.2.

Then, the process of scale analysis is conducted as follows:

p_d = dynamic pressure

x-eqn	$\frac{\partial u}{\partial t}$	$+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}$		$+w\frac{\partial u}{\partial z}$	$=-\frac{1}{\rho}\frac{\partial p_d}{\partial x}$	$+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$	$+\nu\frac{\partial^2 u}{\partial z^2}$	$+2\Omega v\sin\phi$	$-2\Omega w\cos\phi$
y-eqn	$\frac{\partial v}{\partial t}$	$+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}$		$+w\frac{\partial v}{\partial z}$	$=-\frac{1}{\rho}\frac{\partial p_d}{\partial y}$	$+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$	$+\nu\frac{\partial^2 v}{\partial z^2}$	$-2\Omega u\sin\phi$	
scale	$\frac{U^2}{L}$	$\frac{U^2}{L}$		$\frac{UW}{H}$	$\frac{\delta p}{\rho L}$	$\frac{\nu U}{L^2}$	$\frac{\nu U}{H^2}$	$f_0 U$	$\approx f_0 W$
magnitude	10^{-4}	10^{-4}		10^{-5}	10^{-3}	10^{-16}	10^{-12}	10^{-3}	10^{-6}

leading terms

Based on these magnitudes, we can see that the viscous term makes a very small contribution to the force balance experienced in a typical mid-latitude cyclone. So we can ignore that term with very little impact on the precision of our results. In fact, to a good degree of accuracy, we can also discard the Coriolis deflection caused by the vertical motion in the system $2\Omega w\cos\phi$, and the vertical advection term $w\partial u/\partial z$. Hence, we can rewrite the horizontal component of the Navier–Stokes equations in an approximate form:

$$\boxed{\begin{aligned}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}&=-\frac{1}{\rho}\frac{\partial p_d}{\partial x}+2\Omega v\sin\phi\\ \frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}&=-\frac{1}{\rho}\frac{\partial p_d}{\partial y}-2\Omega u\sin\phi\end{aligned}} \quad (5.8)$$

much simplified form

Table 5.2 Typical scales for a mid-latitude weather system

Scale	Symbol	Magnitude
Horizontal wind scale	U	$10\,\mathrm{m\,s^{-1}}$
Vertical wind scale	W	$10^{-2}\,\mathrm{m\,s^{-1}}$
Horizontal length scale	L	$10^{6}\,\mathrm{m}$
Vertical length scale (depth of troposphere)	H	$10^{4}\,\mathrm{m}$
Time scale (L/U)	T	$10^{5}\,\mathrm{s}$
Kinematic viscosity	ν	$10^{-5}\,\mathrm{m^2\,s^{-1}}$
Dynamic pressure scale	$\delta p/\rho$	$10^{3}\,\mathrm{m^2\,s^{-2}}$
Total pressure scale	P/ρ	$10^{5}\,\mathrm{m^2\,s^{-2}}$
Gravity	g	$10\,\mathrm{m\,s^{-2}}$
Density variation scale	$\delta\rho/\rho$	10^{-2}

Pg 5.6

h = horizontal component of $\vec{U} = \vec{U}_h + w\hat{k}$

We can write this in a shorter vector form if we define the horizontal wind to be $\bar{u}_h = u\bar{i} + v\bar{j}$ and the horizontal material derivative to be

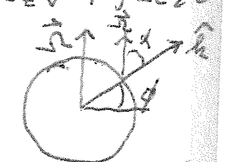
$$\frac{D_h}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

$$\vec{\Omega} \times \vec{U}_h = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Omega_x & \Omega_y & \Omega_z \\ U & V & 0 \end{vmatrix} = -\hat{i}\,\Omega_z V + \hat{j}\,\Omega_z U$$

Then, we can write

$$\frac{D_h u}{Dt} = -\frac{1}{\rho}\frac{\partial p_d}{\partial x} + 2\Omega v \sin\phi$$

$$\frac{D_h v}{Dt} = -\frac{1}{\rho}\frac{\partial p_d}{\partial y} - 2\Omega u \sin\phi$$



$$\Rightarrow \boxed{\frac{D_h \bar{u}_h}{Dt} = -\frac{1}{\rho}\left(\frac{\partial p_d}{\partial x}\bar{i} + \frac{\partial p_d}{\partial y}\bar{j}\right) - 2\bar{\Omega} \times \bar{u}_h}$$

$\Omega_z = \vec{\Omega}\cdot\hat{k}$

$\Omega_z = \Omega\cos\alpha$
$= \Omega\sin\phi$
since $\alpha = \pi/2 - \phi$

or

$\vec{k}$ = unit vector

$$\boxed{\frac{D_h \bar{u}_h}{Dt} = -\frac{1}{\rho}\left(\frac{\partial p_d}{\partial x}\bar{i} + \frac{\partial p_d}{\partial y}\bar{j}\right) - f\bar{k} \times \bar{u}_h} \qquad (5.9)$$

Turning to the vertical momentum equation, and using the total pressure form:

z-eqn	$\frac{\partial w}{\partial t}+$	$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}$	$+w\frac{\partial w}{\partial z}$	$=-\frac{1}{\rho}\frac{\partial p}{\partial z}$	$+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)$	$+\nu\frac{\partial^2 w}{\partial z^2}$	$-g$	$+2\Omega u\cos\phi$
scale	$\frac{UW}{L}$	$\frac{UW}{L}$	$\frac{W^2}{H}$	$\frac{P}{\rho H}$	$\frac{\nu W}{L^2}$	$\frac{\nu W}{H^2}$	g	$f_0 U$
magnitude	10^{-7}	10^{-7}	10^{-8}	10	10^{-19}	10^{-15}	10	10^{-3}

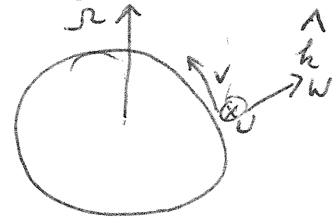
Thus, the atmosphere in motion is strongly hydrostatic on the synoptic scale, since to an excellent degree of accuracy, we can write

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = -g$$

Are the disturbances themselves hydrostatic? We find that if we do the same scale analysis using the dynamic pressure and the buoyancy term, there is a similar balance:

z-eqn	$\frac{\partial w}{\partial t}+$	$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}$	$+w\frac{\partial w}{\partial z}$	$=-\frac{1}{\rho}\frac{\partial p_d}{\partial z}$	$+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)$	$+\nu\frac{\partial^2 w}{\partial z^2}$	$-\frac{(\rho-\rho_0)}{\rho}g$	$+2\Omega u\cos\phi$
scale	$\frac{UW}{L}$	$\frac{UW}{L}$	$\frac{W^2}{H}$	$\frac{\delta p}{\rho H}$	$\frac{\nu W}{L^2}$	$\frac{\nu W}{H^2}$	$\frac{\delta\rho g}{\rho}$	$f_0 U$
magnitude	10^{-7}	10^{-7}	10^{-8}	10^{-1}	10^{-19}	10^{-15}	10^{-1}	10^{-3}

So the answer is yes: synoptic scale disturbances themselves are strongly hydrostatic. This means that vertical velocities in such systems tend to be relatively small and are excited by departures from hydrostatic balance. One must always be careful to distinguish between the hydrostatic *equation* for an atmosphere at rest (Section 4.3.4) and the hydrostatic *approximation* for synoptic scale atmospheric motions derived here.

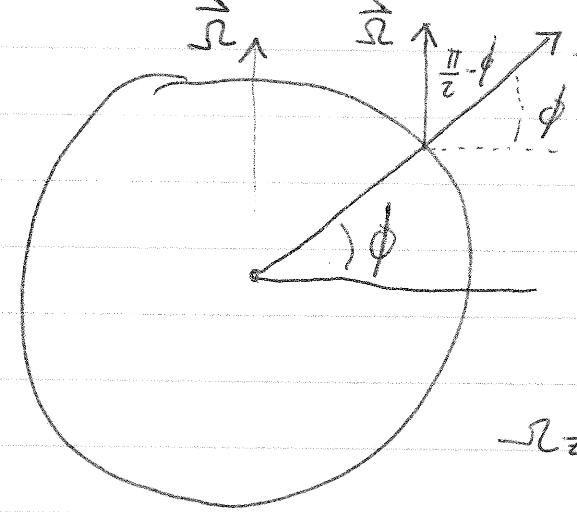


Aside

$$\vec{U}_h = (U, V, 0)$$

$$\vec{\Omega} \times \vec{U}_h = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Omega_x & \Omega_y & \Omega_z \\ U & V & 0 \end{vmatrix}$$

$$= -i\,\Omega_z V - j(-\Omega_z U) = -\Omega_z V\hat{i} + \Omega_z U\hat{j}$$



$$\Omega_z = \vec{\Omega}\cdot\hat{k}$$
$$= \Omega\cos\left(\tfrac{\pi}{2}-\phi\right)$$
$$= \Omega\sin\phi$$

Note:

$$\hat{k} \times \vec{U}_h = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ U & V & 0 \end{vmatrix}$$

$$= -V\hat{i} + U\hat{j}$$

So if $f \equiv 2\Omega\sin\phi$

$$\boxed{-2\vec{\Omega}\times\vec{U}_h = -f\,\hat{k}\times\vec{U}_h}$$

§ 5.5 Geostrophic Approximation :

$$\text{PGF} + \text{COR}$$

$$\hat{x} \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \approx 0 = -\frac{1}{\rho}\frac{\partial P_d}{\partial x} + 2\Omega v \sin\phi$$

$$\hat{y} \quad \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \approx 0 = -\frac{1}{\rho}\frac{\partial P_d}{\partial y} - 2\Omega u \sin\phi$$

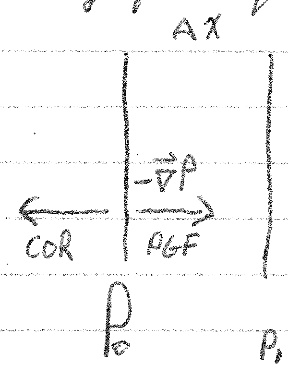
$$\boxed{f = 2\Omega \sin\phi}$$

[Can use scaling to find out when an OK approximation]
[Defines a velocity field for the homogeneous solution of $\hat{x}$ & $\hat{y}$]

Example

$\hat{x}$ Component

$\longrightarrow$ x



Here $\text{PGF} = -\frac{\vec{\nabla}P}{\rho} > 0$

$\text{COR} = fv < 0$

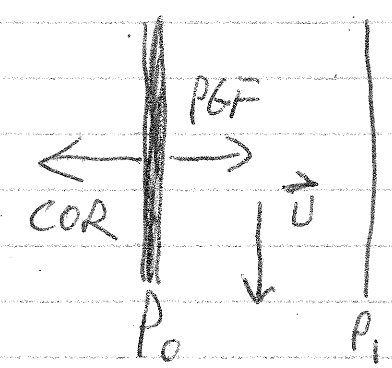
For balance, must have $v < 0$.

$$\text{PGF} = -\text{COR}$$

$$\text{PGF} = \underbrace{\left[-\frac{1}{\rho}\frac{(P_1 - P_0)}{\Delta x}\right]}_{>0} = -fv$$

$$v = -\frac{\text{PGF}}{f}$$

$$v = \left(\frac{P_1 - P_0}{\rho \Delta x f}\right) \text{ is } < 0$$

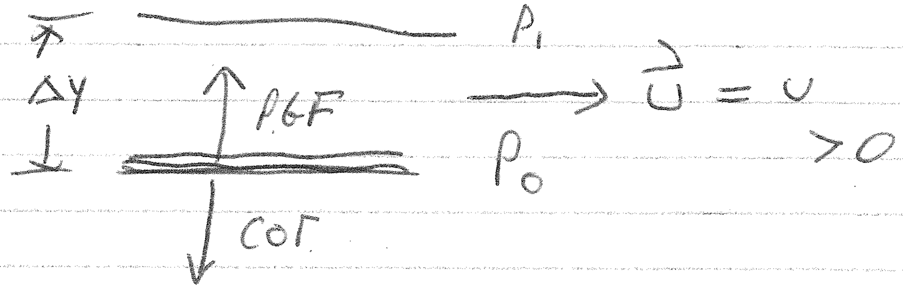


Point left hand to low pressure, right to high. Wind will be at your back.

Rule

use Rule

y component



$$\hat{y} \qquad -\frac{1}{\rho}\frac{\partial p_d}{\partial y} - fu = 0$$

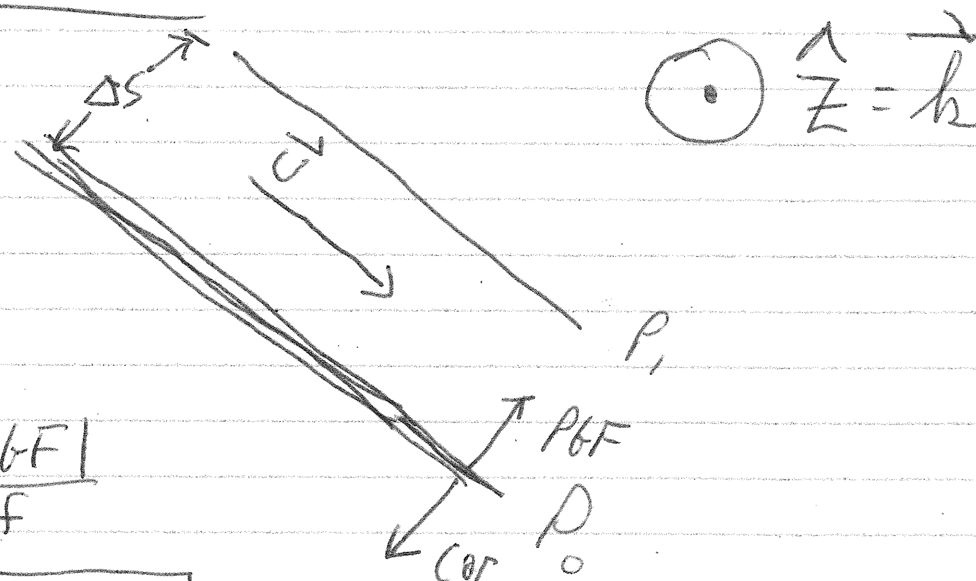
PGF COR $\qquad COR = -fu$

$$PGF > 0 = -\frac{1}{\rho}\left(\frac{P_1 - P_0}{\Delta y}\right) = -COR = fu$$

$$\boxed{u = \frac{P_0 - P_1}{\rho \Delta y f}}$$

Natural Coordinates

Top View



$$|\vec{U}| = \frac{|PGF|}{f}$$

$$\boxed{|\vec{U}| = \frac{|P_0 - P_1|}{\rho \Delta s f}}$$

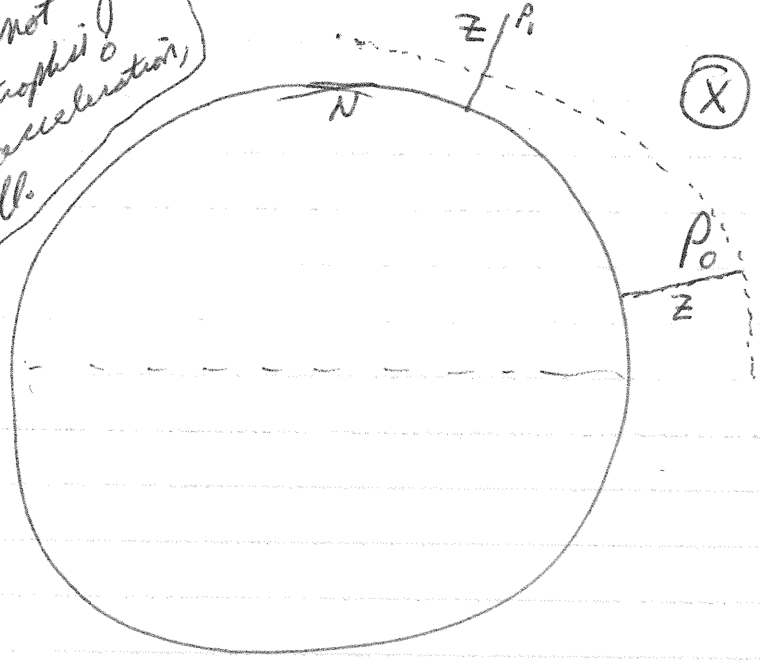
$$\boxed{\vec{U} = -\frac{\vec{k} \times \vec{PGF}}{f}}$$

Direction $\vec{U} = \vec{U}/|\vec{U}|$

Pg 5.10
By Ruelle

(X) Wind direction

Note
Winds are not always geostrophic!
$\partial \vec{U} + \vec{U}\cdot\nabla\vec{U}$ is acceleration,
not always small.



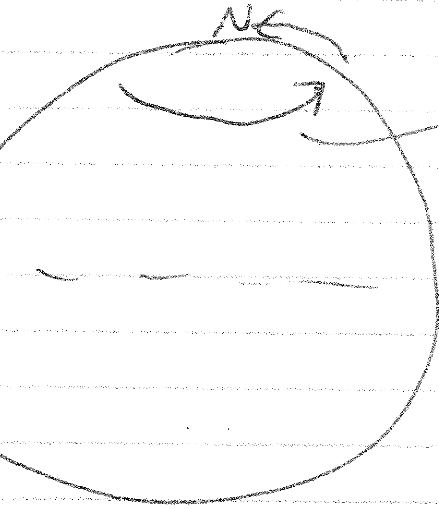
Height of Atmos

Ageostrophic flow
not
gives rise to upper level divergence and convergence
$\Rightarrow$ upper level support for storms!

UP high, $P_0(Z) > P_1(Z)$ for same Z

Near Equator   Near Pole

So expect wind $\vec{U} \sim U\uparrow$
[westerlies]



Winds $\Rightarrow$ Polar vortex