Chapter 2 Math Methods in Fluid Dynamics

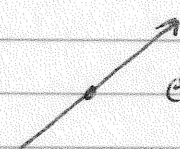
Key: Introduce Lagrangian and Eularian perspectives; Advection, Vorticity, and divergence.

§ 2.1 Scalars and vectors

Scalars - numbers, time, temperature, wind speed, pressure,

Vectors - have a magnitude and direction

Q: What is this wind direction? (235°)



Most common example is wind velocity.

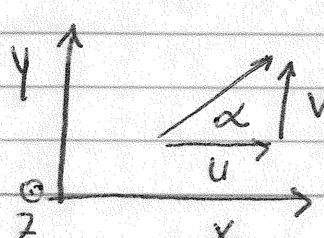
Horizontal wind:

From the North, 0°

From the west 270°

From the East 90°

From the South 180°



$$\vec{u} = u\hat{x} + v\hat{y} = u_x\vec{i} + u_y\vec{j}$$

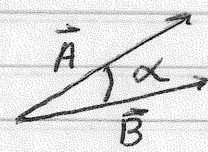
$$\alpha = \mathrm{Tan}^{-1}\, v/u$$

$$[-180, \alpha, 180]$$

Dot Product of Vectors:

is a scalar

<u>Scalar Product</u>



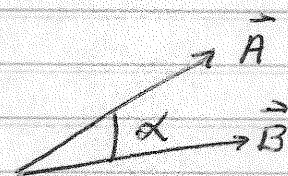
$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z = |\vec{A}| \cdot |\vec{B}| \cdot \cos\alpha$$

Easy way to compute α

Cross Product of 2 vectors:

Is a vector

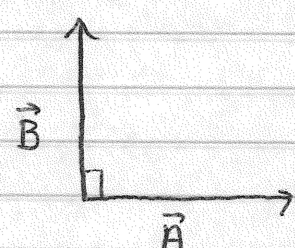
<u>Vector Product</u>



$$\vec{A} \times \vec{B} = |\vec{A}|\,|\vec{B}| \sin\alpha$$

Direction by right hand rule.

<u>Example</u>



$\vec{A} = A_x \hat{x}$

$\vec{B} = B_y \hat{y}$

$\vec{A} \times \vec{B} =$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & 0 & 0 \\ 0 & B_y & 0 \end{vmatrix} = \hat{i}(0 - 0 \cdot B_y) + \hat{j}(A_x \cdot 0 - 0) + \hat{k}(A_x B_y)$$

$$= \hat{k} A_x B_y$$

is perpendicular to both $\vec{A}$ and $\vec{B}$.

§ 2.3 Scalar Field $T(x, y, z)$

$P(x, y, z)$

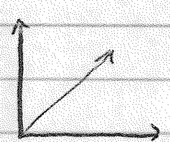
Each position in space

Vector Field $\vec{U} = \vec{U}(x, y, z, t)$

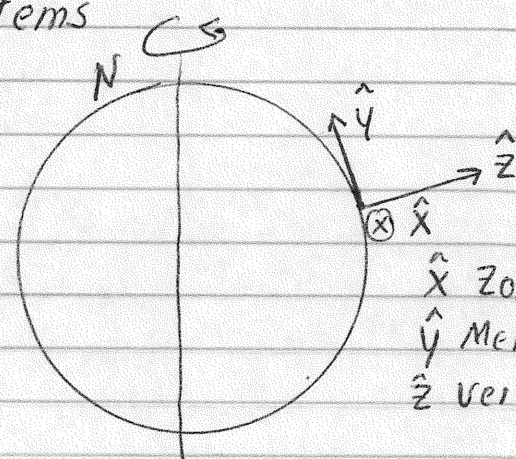
$$\vec{U} = U(x,y,z,t)\,\hat{x} + V(x,y,z,t)\,\hat{y} + W(x,y,z,t)\,\hat{z}$$

Scalar Fields are independent of coordinate system.

§ 2.4 Coordinate Systems

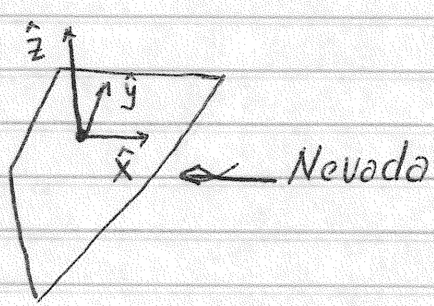


Fixed Coordinate system in space [inertial]



$\hat{x}$ Zonal
$\hat{y}$ Meridional
$\hat{z}$ vertical

Rotating Coordinate System constanty changing, non inertial, accelerating.



§2.5 Gradients of vectors

Consider $\vec{U}$ = velocity of wind

$= u(x,y,z,t)\,\hat{x} + v(x,y,z,t)\,\hat{y} + w(x,y,z,t)\,\hat{z}$

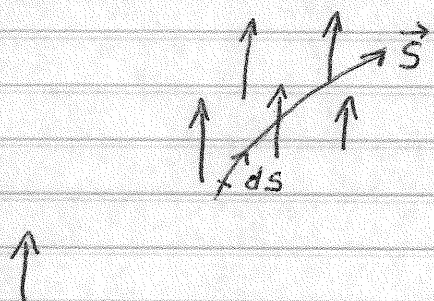
The acceleration vector is [Partial derivative with respect to time]

$$\vec{a} = \frac{\partial \vec{U}}{\partial t} = \frac{\partial u}{\partial t}\,\hat{x} + \frac{\partial v}{\partial t}\,\hat{y} + \frac{\partial w}{dt}\,\hat{z}$$

§2.6 Line integrals: Work and potential energy change

Line integral example:

$$I = \int_a^b f(x)\,dx$$



$\vec{s}$ = Path of air Parcel through force field

Force Field due to pressure gradient.

Work $= \int \vec{F} \cdot d\vec{s}$ = Work done on air parcel by $\vec{F}$

$$W = \int_{\vec{s}} (F(x,y,z,t)\hat{\imath} + g\hat{\jmath} + h\hat{k}) \cdot (\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz)$$

$$= \int_{\vec{s}} F\,dx + g\,dy + h\,dz$$

We may always derive f, g, h
from a scalar function iff

$$f\,dx + g\,dy + h\,dz \text{ is an}$$

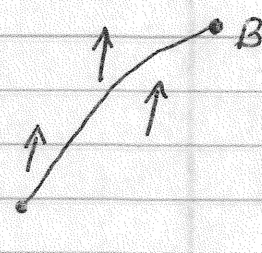
exact differential: Let $\phi(x,y,z,t)$ = Potential function of $\vec{F}$

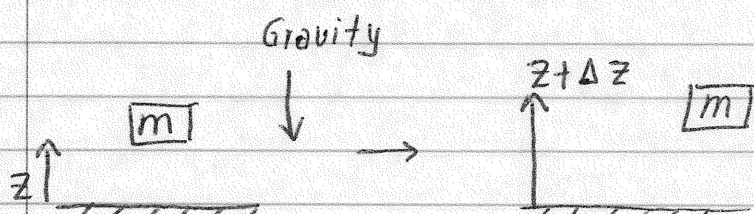
$$f(x,y,z,t) = \frac{\partial \phi}{\partial x}$$

$$g = \frac{\partial \phi}{\partial y} \; ; \quad h = \frac{\partial \phi}{\partial z}$$

Then $W = \int_{\vec{s}} (f\,dx + g\,dy + h\,dz)$

$$= \int_{\vec{s}} \frac{\partial \phi}{\partial s}\,ds = \phi(B) - \phi(A)$$

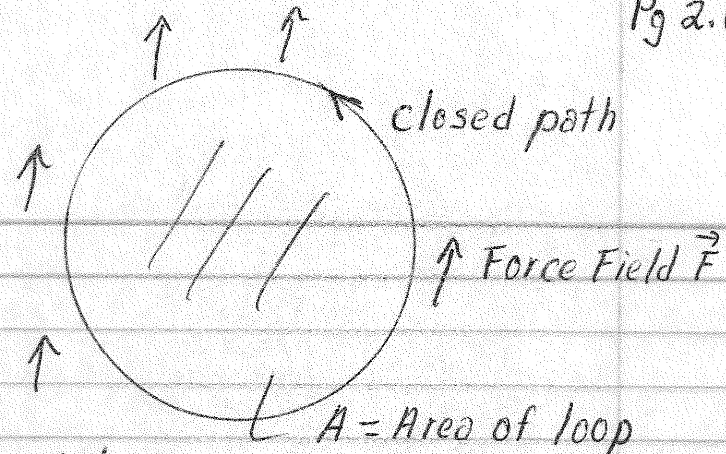




Work $= mg\Delta z$

$\phi = mgz$ = Potential Energy Function

Simple Familiar Example



$\oint$ 2.6.2

Circulation in a Closed Loop

$$\oint \vec{F} \cdot d\vec{s} = \oint (F dx + g dy) = \iint_A \left(\frac{dg}{dx} - \frac{\partial f}{\partial y} \right) dA$$

closed path

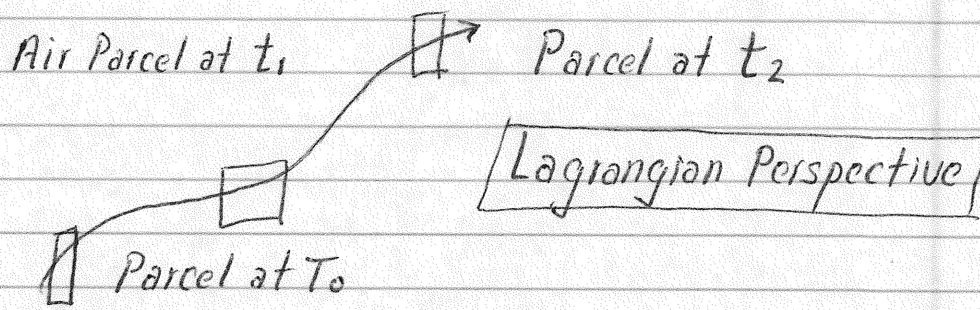
Stokes Theorem

In 3D $\oint \vec{F} \cdot d\vec{s} = \iint_A (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, dA$

Normal to A

We will use circulation and $\vec{\nabla} \times \vec{u}$ to get fluid vorticity.

§2.7.7 Eularian and Lagrangian Frames of Reference



Path line of air parcel, like what is used in back trajectory analysis:
We follow the air parcel and record its properties.
Follow many air parcels to characterize the fluid.

P
$T(x,y,z,t)$ etc. $\qquad \vec{U} \longrightarrow$

$\longrightarrow \quad \bullet\ x,y,z,t$

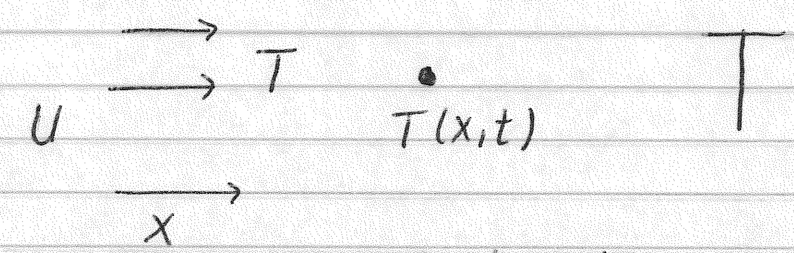
$\uparrow$ Fixed position of observation [Eularian description]

Velocity Field: $\vec{U} = u(x,y,z,t)\hat{i} + v\hat{j} + w\hat{k} = (u,v,w)$

We use the Eularian description and can integrate to get the path line,

$$\vec{S}(t) = \vec{S}_0(t_0) + \int_{t_0}^{t} \vec{u}[\vec{s}(\tau)]\, d\tau$$

§ 2.8 Advection:



Size of symbol represents local temperature value at locations downwind and upwind of x.

Example of cold air advection. In time dt, fluid from $dx = u\,dt$ is brought from left to right causing warmer air to be replaced by colder air.

$$\Delta T = T - \mathbf{T} \quad \leftarrow \text{Initial air } T \text{ at } x$$

↑ Final air T at x after wind blows for time dt

$$= T(x-dx) - T(x)$$

$$\simeq T(\cancel{x}) - \frac{\partial T}{\partial x}\,dx - T(\cancel{x})$$

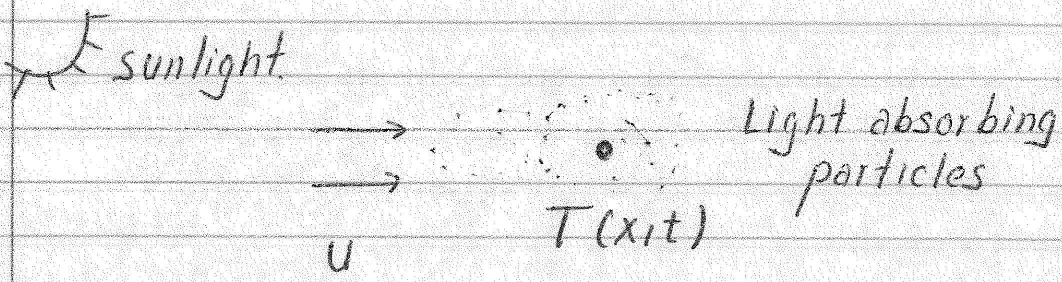
$$= -u\,dt\,\frac{\partial T}{\partial x}$$

So

$$\boxed{\frac{\partial T}{\partial t} = -u\frac{\partial T}{\partial x}}$$

Temperature change at x due to advection.

Total temperature change at x includes effect of other local mechanisms like heating

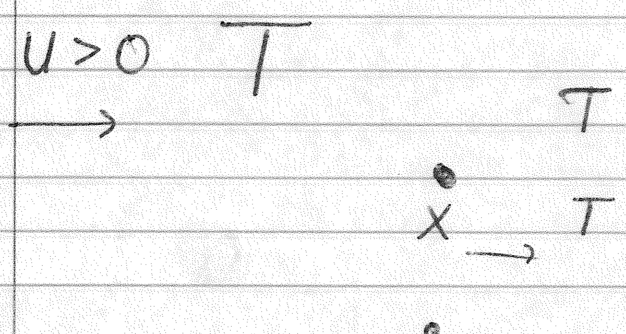


Absorption of sunlight causes heating as well as the air parcel moves along:

$$\left.\frac{DT}{Dt}\right) \text{along parcel}$$

So total temperature change is $\dfrac{\partial T(x,t)}{\partial t} = \dfrac{DT}{Dt} - U\dfrac{\partial T}{\partial x}$ "Temperature Advection by wind U"

In situ heating or cooling



$$\frac{\partial T}{\partial x} < 0$$

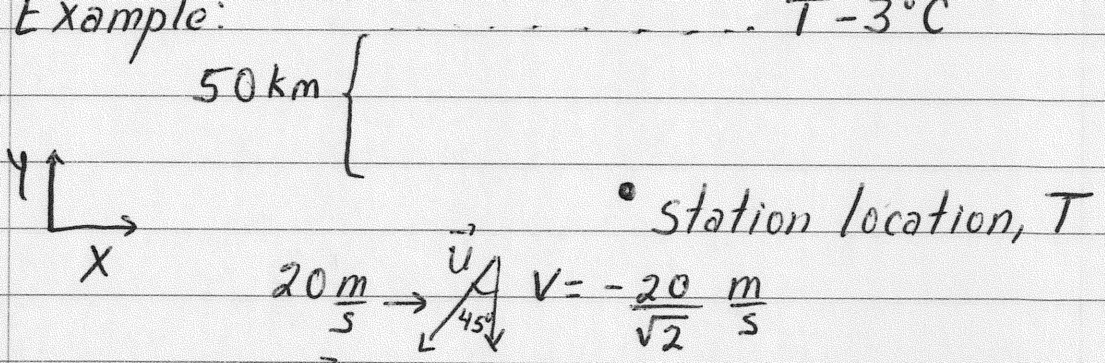
Warm Advection

Generally,

$$\frac{\partial T}{\partial t} = \frac{DT}{Dt} - \vec{U} \cdot \vec{\nabla} T$$

$$\vec{U} = (u, v, w) \; ; \quad \vec{\nabla} T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right)$$

Example:



Advection brings in colder air.
Radiation heats at a rate of 1°C/hr = 1°C / 3600 secs
What is the rate of T change?

<u>Advection Temperature change</u>

$$\frac{\partial T}{\partial t} = -V \frac{\partial T}{\partial y} = -\left(\frac{-20}{\sqrt{2}} \frac{m}{s} \right) \left(\frac{-3°C}{50{,}000 m} \right)$$

$$\left(\frac{\partial T}{\partial t} \right)_{\text{Advection}} = -\frac{1}{1179} \frac{°C}{sec} \sim -\frac{3°C}{hr}$$

$$\boxed{\frac{\partial T}{\partial t} = \frac{DT}{Dt} - V \frac{\partial T}{\partial y} = 1 \frac{°C}{hr} - 3 \frac{°C}{hr} = -2 \frac{°C}{hr}}$$