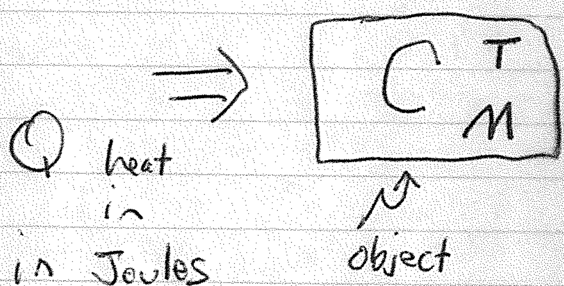
Example Problem

Problem Statement:

Given CO_2 forcing is about $3\ w/m^2$; how long would it take to raise the temperature of the atmosphere by $1\ °C$? If this power was instead dropped into the ocean, how long would it take to raise ocean temperature by $1\ °C$?
Note: $3\ w/m^2$ is the "extra" radiative forcing caused by Anthropogenic CO_2 emitted to the atmosphere since the industrial revolution.

Solution:

Let's treat the problem in a general way.



C = Heat capacity / mass $\left[\frac{J}{K\ kg}\right]$ ("Specific heat")

M = Object mass

ΔT = temperature change

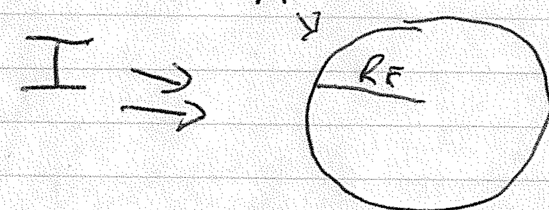
$$Q = \cancel{\Delta T} = MC\Delta T$$

(1) $$\boxed{\Delta T = Q/MC}$$

We need to convert radiative forcing to Q.

Let $I \equiv 3 \frac{W}{m^2}$ = irradiance to be converted to heat.

A = area radiation absorbed over



Δt = time absorbtion occurs.

(2) Then $Q = I\left[\frac{W}{m^2}\right] * A\,[m^2] * \Delta t$

since $W = \frac{J}{s}$.

Now all we need to do is to find the mass of the atmosphere and ocean and the area A.

Combining (1) and (2) we can get the time needed for the temperature change by 1 K.

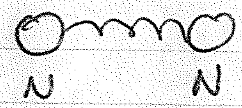
$$\Delta T = \frac{I A \Delta t}{m C}$$ So the

(3) time needed is $\boxed{\Delta t = \frac{m C \Delta T}{I A}}$

A high heat capacity C and mass M makes for a long time to increase the temperature by 1 C.

Aside: Heat capacity per unit mass for air and water.

Air is composed of N_2 and O_2, mostly.



diatomic molecule

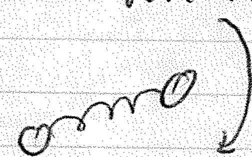
From the equipartition theorem of physics, the energy of air is, for N molecules,

(4) $$E = N n \frac{1}{2} kT$$

n = "# degrees of freedom, energy pathways for a molecule"

½ kT energy for every mode of vibration or rotation or translation.

rotation



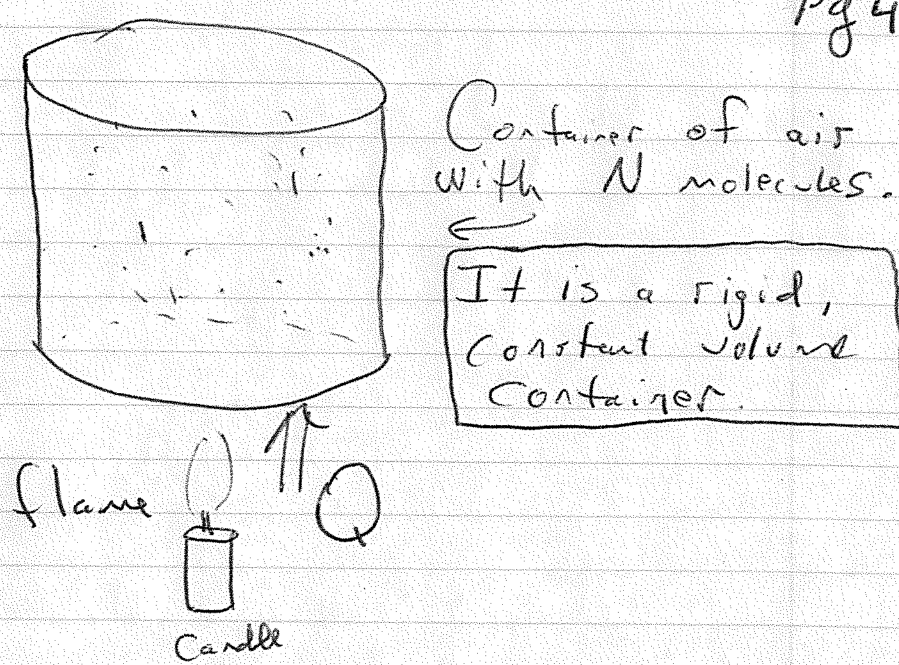
Can move in 3 directions,

Can rotate in 2 direction.

Vibration is "frozen out" at atmospheric temperatures.

n = 5 for N_2 and O_2 molecules

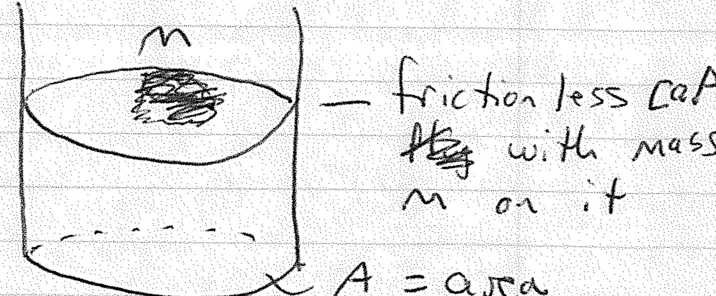
In (4), $k = 1.38 \times 10^{-23} \frac{J}{\text{molecule } ^\circ K}$ is Boltzmann's constant.



When we add Q heat into our air sample the energy change will be

(5) $$\Delta E = Q = \frac{5}{2} N k_B T \, .$$

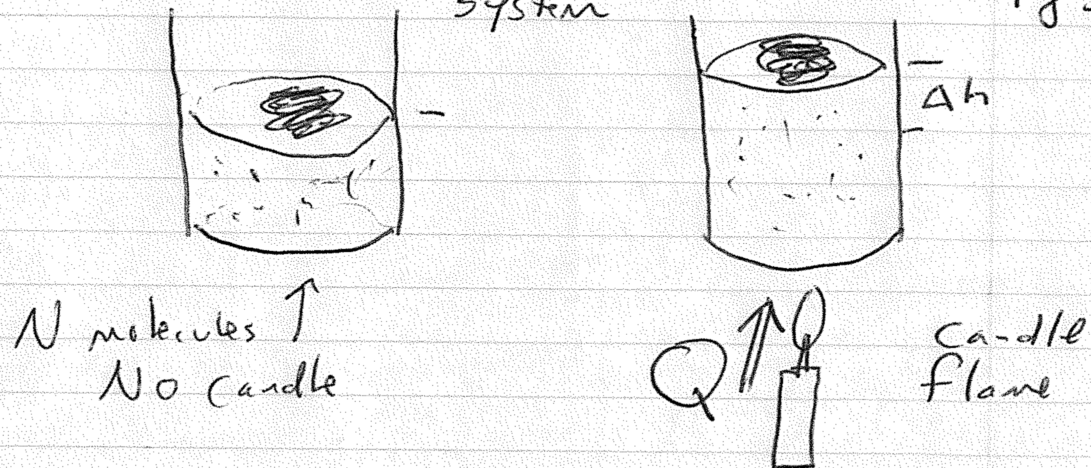
That's fine, but the atmosphere is not a sealed container. It is more like this



When we apply the candle, the cap can move up.

(6) The pressure in the chamber is $\frac{mg}{A} = P$

Constant Pressure System



The candle both causes the molecules to move around faster, and the cap to move up. The increase in potential energy of the cap is

(7) $$P.E. = mg\Delta h .$$

This increase in P.E. is provided by the action of the faster moving molecules pushing the cap up; they were made to move faster by the heat provided by the candle flame. We can play with Eq (7) to get

(8) $$P.E. = \frac{mg}{A} A\Delta h$$

The effect of Q is

(9) $$Q = \frac{5}{2} N k \Delta T + P \Delta V$$

internal energy of the gas molecules has gone up, but some of Q goes into the work done by the gas, $P\Delta V$, to move the cap up.

In the constant volume system on page 4 and Eq (5), all of Q goes into changing the temperature, while in the constant pressure system of page 5 and Eq 9, some of Q goes into doing work, changing the gas volume, so the temperature change in the constant pressure system is less than in the constant volume system.

Our ideal gas air equation is

(10) $$PV = NkT$$ so if we hold pressure constant, and ask, how much do we need to change the temperature of the gas by to ~~increase~~ do the work $P\Delta V$ in Eq (9), we find out quickly that

(11) $$P\Delta V = Nk\Delta T \, .$$

So for the same amount of heat Q from the candle, we will have a different temperature change for our constant volume and constant pressure systems.

(9) $$Q = \frac{7}{2} N k \Delta T$$ constant pressure

(5) $$Q = \frac{5}{2} N k \Delta T$$ constant volume

We define:

(12) $$C_p = \frac{7}{2} N k$$

(13) $$C_v = \frac{5}{2} N k$$

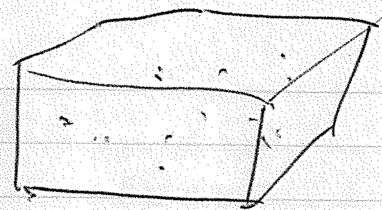
as the heat capacity for constant pressure and constant volume processes.

Then

(9) $$Q = C_p \Delta T$$ constant pressure

(5) $$Q = C_v \Delta T$$ constant volume

Next we want to work of the specific heat capacity $c_p = C_p / \text{mass}$, the heat capacity per unit mass of air.



← Box of air with N molecules.

M = mass of air in Box. What is the mass?

$$M = N_{\text{molecules}} * \mu \frac{kg}{mole} * \frac{mole}{N_a \text{ molecules}}$$

μ ← Average Molecular weight of air

N_a ← Avagadro's #

Rearranging,

(14) $$N = \frac{M N_a}{\mu} \text{ molecules}$$

Using (14) in (12) ~~and (13)~~,

$$C_p = \frac{7}{2} N k = \frac{7}{2} \frac{M}{\mu} \underbrace{N_a k}_{R^*}$$

(15) $$C_p = \frac{7}{2} \frac{M}{\mu} R^* = \frac{7}{2} M \underbrace{\frac{R^*}{\mu}}_{R_{air}} .$$

We used a universal gas constant definition,

$$R^* = N_a k$$

and

$$R_{air} = R^* / \mu .$$

We get $$c_p = \frac{C_p}{M} = \frac{7}{2} R_{air} .$$

The value of R_{air} is

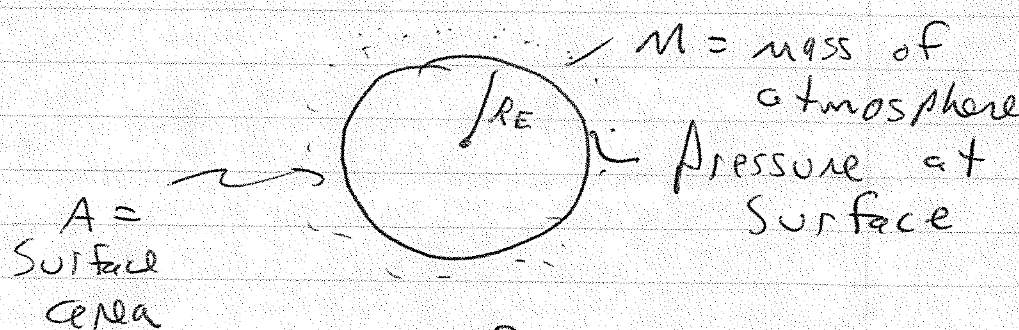
$$R_{air} = \underbrace{8.3143 \frac{J}{mole\ K}}_{R^*} \; \underbrace{\frac{1000\ mole}{29\ Kg}}_{\mu}$$

(16) $$R_{air} = 286.7 \frac{Joules}{Kg\ °K}.$$

Now we return to our problem, calculating Δt in Eq 3, the time it takes for the atmosphere to increase temperature by 1 K due to absorption of radiation I;

(3) $$\Delta t = \frac{M}{A} \frac{C_p \Delta T}{I}.$$

Referring back to Eq (6), the surface pressure on the planet is



(6) $$P = \frac{Mg}{A}. \text{ So}$$

(17) $$\Delta t = \frac{P}{g} \frac{C_p \Delta T}{I}.$$

For the atmosphere then,

$$\Delta t = \frac{\frac{10^5\ Pa}{9.8\ m/s^2} \cdot \frac{7}{2} \cdot 286.7 \frac{J}{Kg\,K} \quad \underset{\sim}{\Delta T}\ 1K}{3\ W/m^2}$$

$\Delta t = 3.4 \times 10^6$ Seconds

$\boxed{\Delta t = 39.5 \text{ days}}$ Atmosphere.

Given about 40 days and an excess radiative forcing of 3 w/m³, the atmosphere will warm up, on average, by 1 °C.

For the oceans, we use

$m = 1.38 \times 10^{21}$ Kg (mass of oceans)

M = mass of oceans

$A = 4\pi R_e^2 = 4\pi \times (6.37 \times 10^6\ m)^2$

$A = 5.1 \times 10^{14}\ m^2$

$C = 1 \frac{\text{calorie}}{\text{gram}\ ^\circ K} = 4180 \frac{J}{Kg\,K}$

$\Delta t = \frac{m}{A} \frac{C \Delta T}{I} = \frac{1.38 \times 10^{21}}{5.1 \times 10^{14}} \times \frac{4180 \quad 1K}{3}$

$\Delta t = 3.8 \times 10^9$ seconds

$= 43{,}636$ days

$\boxed{\Delta t = 120 \text{ years}}$

In summary, the atmosphere takes about 40 days to warm by 1 K due to excess radiation absorbed of $3 \frac{W}{m^2}$ while the oceans take about 120 years to warm by 1 K.

Roughly speaking, the atmosphere responds quickly to radiative imbalances, though oceans take much longer; about a factor of

$$\frac{43636}{40} \frac{\text{days}}{\text{dys}} \frac{\text{ocean}}{\text{atmosphere}}$$

~ factor of 1000 times as long for ocean to heat up as atmosphere.

The bad news is, eventually the ocean does warm up too.

Fun thoughts in closing

For air, $C_p = \frac{7}{2} R_{air} = 1004 \frac{J}{Kg\,K}$

For water $C = 4180 \frac{J}{Kg\,K}$

Per mole,

Air $C_p = 1004 \frac{J}{KgK} * \frac{29}{1000} \frac{Kg}{mole} \approx 29 \frac{J}{mole\,K}$

$C = 4180 \frac{J}{KgK} * \frac{18}{1000} \frac{Kg}{mole} = 75 \frac{J}{mole\,K}$

The heat capacity of $\frac{\text{water}}{\text{air}} \approx 2.6$

So Liquid water molecules have about 2.6 equivalent degrees of freedom more than the N_2 and O_2 molecules of air.

For water vapor in the atmosphere

$n = 6$ (3 rotation + 3 translation)

degrees of freedom so

$$C_p^{\text{water vapor}} = 4 R^* \frac{J}{\text{mole } ^\circ K} \approx 33 \frac{J}{\text{mole } K}$$

which is considerably less than the $\frac{\text{heat capacity}}{\text{mole}}$ of water molecules in the liquid state.